SELF LEARNING MATERIAL

Master of Arts/Science

MATHEMATICS

COURSE: MATH - 204

INVISCID FLUID MECHANICS

BLOCK - 1, 2, 3, 4 & 5

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PREFACE

Fluid dynamics is the science of flow of fluids. The science of Fluid dynamics covers a vast area and it is imposible to described its basic fully within the covers of a single book. It stretches accross a wide range of natural phenomena from the swimming of microorganisms at one end to the evolution of galaxies at the other. A good idea of the range of topics can be obtained by a mere persual of the title of articles of published every year in the Annual Reviews of Fluid Mechanics.

BLOCK - 1

KINEMATICS

Structure :

1.0 **Objective** :

The main concern of the Science of Fluid is to study the motion of fluids or that of bodies in contact with fluid. Fluid dynamics occupies an important place in modern science and engineering. It forms one of the foundation of aeronautics and astronautics, mechanical engineering, meteriology, marine engineering, civil engineering, bioengineering and in fact just about every scientific or engineering field.

In this unit basic concepts and definitions related to fluid and its properties are discussed. Stream line, path line and streak line are defined and the differential equations whose solutions yield these lines are obtained. The condition to be satisfied by a boundary surface are derived. The equation of continuity, the equations of motion and the equation of energy are formulated. The concept of circulation has also been given.

1.1 **Introduction** :

A fluid is a material which flows. Fluids are classified as liquids and gases. Liquids are not sensibily compressible except under the action of heavy forces whereas gases are easily compressible, and expand to fill any closed space. A liquid at constant temperature and pressure has a definite volume and when placed in an open vessel will take, under the action of gravity, the form of the lower part of the vessel and will be bounded above by a horizontal free surface. All known liquids are to some slight extent compressible. For most purposes it is, however, sufficient to regard liquids as incompressible fluids.

1.2 Definitions and Basic Concept :

It is well known that matter is made up of molecules or atoms which are always in a state of random motion. In fluid dynamics the study of individual molecules is neither necessary nor appropriate from the point of view of use of mathematical methods. Hence, we consider macroscopic behaviour which treat a fluid as having continuous structure and so at each point we can prescribe a unique velocity, a unique density, etc. The assumption of continuous distribution of fluid in space is known as the continuum hypothesis. This continuum concept of matter allows us to subdivide a fluid element indefinitely. Furthermore, we define a fluid particle as the fluid contained within an infinitesimal volume whose size is so small that it may be regarded as a geometrical point.

An infinitesimal fluid element is acted upon by two types of external forces, namely, body forces and surface forces. Body forces are those which act on an element of mass of a body and are proportional to the mass of the body on which they act. A typical example of body forces is provided by gravitational forces. Surface forces are those which arise at points of the body surface and are proportional to the surface area. Surface forces may be resolved into two components, one normal and the other tangential to the surface upon which they act. The normal component of the force per unit area is called the normal stress or pressure and the tangential component of the force per unit area is called the shearing stress. When an external force is applied to a solid, deformation is produced in the solid. If this force per unit area, viz. stress is less than the yield stress. the deformation disappears when the applied force is removed. If the applied stress is more than the yeild stress, of the material, it acquires a permanent setting or even breaks. If a shearing force is applied to a fluid, it deforms continuously as long as the force is acting on it, regardless of the magnitude of the force. Thus the yield stress for fluids is taken to be zero. A fluid is said to be viscous or real when normal as well as shearing stresses exist. On the other hand, a fluid is said to be non-viscous (or inviscid or perfect or ideal) when it does not exert any shearing stress, whether at rest or in motion. Clearly the pressure exerted by an inviscid fluid on any surface is always along the normal to the surface whereas the pressure at a point in a stationary fluid is independent of direction. Due to shearing stress a viscous fluid produces resistance to the body moving through it as well as between the particles of the fluid itself. Water and air are treated as inviscid fluid whereas glycerene, syrup and heavy oils treated as viscous fluids. It is, however, to be pointed out that it is, only imaginary situation where the fluid is assumed to be inviscid and incompressible.

The density of a fluid is defined as the mass per unit volume. Mathematically the density ρ at a point P may be define as -

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v}$$

where δv is the volume element around the point P and δm is the mass of the fluid within the volume δv.

1.3 **Lagrangian and Eulerian Methods** :

There are two methods of treating the problems of fluids dynamics mathematically : (i) Lagrangian method and (ii) Eulerian Method. In Lagrangian method we study the history of each fluid particle *i.e.*, any fluid particle is selected and is pursued on its onward course observing the changes in velocity, pressure and density at each point and at each instant. Let (x_0, y_0, z_0) be the coordinates of the chosen particle at a given time t=t_0. At a later time t=t, let the coordinates of the same particle be (x,y,z). Since the chosen particle is any particle in the fluid, the coordinates (x,y,z) will be functions of t and also of their initial values (x_0,y_0,z_0), so that

$$x = f_1(x_0, y_0, z_0, t), y = f_2(x_0, y_0, z_0, t), z = f_3(x_0, y_0, z_0, t)$$

Let u,v,w and a_x,a_y,a_z be the components of velocity and acceleration respectively. Then we have

$$u = \frac{\partial x}{\partial t},\; v = \frac{\partial y}{\partial t},\; w = \frac{\partial z}{\partial t}$$

and $$a_x = \frac{\partial^2 x}{\partial^2 t},\; a_y = \frac{\partial^2 x}{\partial^2 t},\; a_z = \frac{\partial^2 x}{\partial^2 t}$$

In Eulerian method we select any point fixed in the space occupied by the fluid and study the changes which take place in velocity, pressure and density as the fluid passes through this point. Let u,v,w be the components of velocity at the popint (x,y,z) at time t. Then, we have

$$u=g_1(x,y,z,t),\; v=g_2(x,y,z,t),\; w=g_3(x,y,z,t)$$

In this method x,y,z,t are four independent variables and all other quantities are their functions. If we regard (x,y,z) as a fixed point, then the values of u,v,w will tell us what happens at that point as t changes; and if we regard t as fixed, then since (x,y,z) may be any point of the fluid, u,v,w will tell us what is happening at every point of the fluid at the particular instant under consideration.

If we wish to connect the Eulerian and Lagrangian methods in any particular problem, we regard u,v,w as the components of velocity of the fluid particle at the point (x,y,z) and the relation between the two sets of symbols

are then $u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}, \quad w = \frac{dz}{dt}$

1.4 Velocity, Local rate of change and Particle rate of change :

Velocity :

Let a fluid particle be at the point P at time t and let the same fluid particle be at the point Q at time $t+\delta t$ such that

$$\vec{OP}=\vec{r} \text{ and } \vec{OQ}=\vec{r}+\delta\vec{r}$$

Then in the interval of time δt the movement of particle is $\vec{PQ}=\delta\vec{r}$ and hence the particle velocity $\vec{q}$ at the point P is

$$\vec{q}=\lim_{\delta t\to 0}\frac{\delta\vec{r}}{\delta t}$$

$$=\frac{d\vec{r}}{dt}$$

$$=\vec{i}\frac{dx}{dt}+\vec{j}\frac{dy}{dt}+\vec{k}\frac{dz}{dt}$$

$$=\vec{i}\,u+\vec{j}\,v+\vec{k}\,w$$

where $\vec{i},\vec{j},\vec{k}$ are unit vectors in the directions of x,y,z respectively.

Thus the velocity $\vec{q}$ is a function of $\vec{r}$ and t, say

$$\vec{q}=f\left(\vec{r},t\right)$$

If the form of the function f is known, we know the motion of the fluid. If the fluid velocity and all fluid properties say density, pressure, temperature etc., together with the conditions associated with the motion of the fluid are independent of the time so that the flow pattern remains unchanged with respect to the time, the motion is said to be steady. In case of steady motion, in terms of mathematics, derivative of any fluid property with respect to time will be zero. If the fluid properties and conditions at a given point depend not only on the position of the point but also the time, the flow is said to be unsteady.

Material, Local and Convective Derivatives :

Let the fluid in motion be asociated by a scalar point function $\phi(x,y,z,t)$ or $\phi\left(\vec{r},t\right)$. Keeping the point P (x,y,z) fixed, the change in ϕ during an interval of time δt is

$$\phi(x,y,z,t+\delta t)-\phi(x,y,z,t) \text{ or } \phi\left(\vec{r},t+\delta t\right)-\phi\left(\vec{r},t\right)$$

Hence the local time rate of change $\frac{\partial\phi}{\partial t}$ is given by -

$$\frac{\partial\phi}{\partial t}=\lim_{\delta t\to 0}\frac{\phi\left(\vec{r},t+\delta t\right)-\phi\left(\vec{r},t\right)}{\delta t}$$

where $\delta\vec{r}$ is the change in the position of the fixed particle during the short time δt. Therefore

$$\frac{d\phi}{dt}=\lim_{\delta t\to 0}\frac{\phi\left(\vec{r}+\delta\vec{r},t+\delta t\right)-\phi\left(\vec{r},t\right)}{\delta t}$$

gives the individual time rate of change.

Let $\vec{q}(u,v,w)$ be the velocity of the fluid particle, such that

$$\vec{q}=\vec{i}\,u+\vec{j}\,v+\vec{k}\,w \text{ and } \frac{dx}{dt}=u \text{ etc.}$$

then $$\frac{d\phi}{dt}=\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x}\frac{dx}{dt}+\frac{\partial\phi}{\partial y}\frac{dy}{dt}+\frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

$$=\frac{\partial\phi}{\partial t}+u\frac{\partial\phi}{\partial x}+v\frac{\partial\phi}{\partial y}+w\frac{\partial\phi}{\partial z}$$

$$=\frac{\partial\phi}{\partial t}+\left(\vec{i}\,u+\vec{j}\,v+\vec{k}\,w\right).\left(\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$\frac{d\phi}{dt}=\frac{\partial\phi}{\partial t}+\vec{q}.\nabla\phi$$

Similarly for a vector point function $\vec{F}\left(\vec{r},t\right)$ associated with some fluid property (say, velocity) we can show that

$$\frac{d\vec{F}}{dt}=\frac{\partial\vec{F}}{\partial t}+\vec{q}.\nabla\vec{F} \qquad ....(1.1)$$

thus for both scalar and vector point functions we have established operational equivalance.

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\vec{q}.\vec{\nabla} \qquad ....(1.2)$$

applicable to both scalar and vector functions of position and time, provided that these functions are associated with the properties of the moving fluid. $\frac{d}{dt}$ is called the material (or particle or subtantial) derivative, or derivative following the motion of the fluid. The term $\vec{q}.\vec{\nabla}$ is called the convective derivation and it is associated with the change of a physical quantity ϕ or $\vec{F}$ due to the motion of the fluid particle. $\frac{\partial}{\partial t}$ is known as Local derivative.

1.5 **Acceleration of Fluid at a point** :

Let a fluid particle be travelling along a curve. At time t let its position on the curve be $P\left(\vec{r}\right)$ and its velocity $\vec{q}$; along the tangent at the point P to the curve be in the direction of the motion of the particle. Then the instantaneous acceleration $\vec{f}$ at the point P is obtained by replacing $\vec{F}$ by $\vec{q}$ in (1.1) as

$$\vec{f} = \frac{d\vec{q}}{dt} = \frac{\partial \vec{q}}{\partial t} + \left(\vec{q}.\nabla\right)\vec{q} \qquad \text{.......... (1.3)}$$

The expression $\left(\vec{q}.\nabla\right)\vec{q}$ can be written as $\vec{\nabla}\left(\tfrac{1}{2}\vec{q}^{2}\right) - \vec{q}\,x\left(\vec{\nabla} x \vec{q}\right)$

Hence, the expression for acceleration vector $\vec{f}$ given by (1.3) can also be written as

$$\vec{f} = \frac{d\vec{q}}{dt} + \vec{\nabla}\left(\tfrac{1}{2}\vec{q}^{2}\right) - \vec{q}\,x\left(\vec{\nabla}\,x\,\vec{q}\right) \qquad \text{.......... (1.4)}$$

In cartesian co-ordinate system the components f_x, f_y and f_z of acceleration $\vec{f}$ in the direction of x,y and z are respectively given by -

$$\left.\begin{aligned} f_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\ f_y &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\ \text{and} \quad f_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \end{aligned}\right\} \text{........(1.5)}$$

where u,v,w are the components of the velocity in the directions of x,y,z respectively.

In cylinrical polar coordinates (r, θ,z) the components of the acceleration in the directions of r, θ and z are respectively given by--

$$\left.\begin{aligned} f_r &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} \\ f_\theta &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} \\ \text{and } f_z &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial v}{\partial z} \end{aligned}\right\} \text{........(1.6)}$$

where u,v,w are the velocity components in the directions of r, θ, z respectively.

In spherical polar co -ordinates (r, θ, z) the components of the acceleration in the directions of r, θ, ϕ are respectively given by-

$$\left.\begin{aligned} f_x &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial u}{\partial \phi} - \frac{v^2+w^2}{r} \\ f_\theta &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial v}{\partial \phi} - \frac{w^2\cot\theta}{r} + \frac{uv}{r} \\ f_\phi &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + \frac{w}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{uw}{r} + \frac{vw\cot\theta}{r} \end{aligned}\right\} \text{........(1.7)}$$

where u,v,w are the velocity components in the directions of r, θ, ϕ respectively .

1.6 Path lines, stream lines and Streak lines :

If we fix our attention on a particular fluid particle then the curve which this particle describes during its motion is called its path line.

Let the coordinate of a fluid particle be $\vec{\xi}$ at the time of start (t=o) and its coordinate after the time t be $\vec{X}$, then $\vec{X}$ is a function of t and $\vec{\xi}$, i.e.

$$\vec{X} = \vec{X}\left(\vec{\xi}, t\right), X^i = X^i\left(\xi^1, \xi^2, \xi^3, t\right) \text{ .... (1.8)}$$

The initial coordinate $\vec{\xi}$ of a particle is called as material coordinate or convected coordinate and the coordinate $\vec{X}$ as spacial coordinate. Motion is assumed to be continuous so that (1.8) can be inverted as

$$\vec{\xi} = \vec{\xi}\left(\vec{X}, t\right), \xi^i = \xi^i\left(X^1, X^2, X^3, t\right) \quad \text{......................} \ (1.9)$$

If the velocity is given in the spacial coordinates, the solution of the differential equations

$$\frac{d\vec{X}}{dt} = \vec{q}\left(\vec{X}, t\right) \text{ or } \frac{d\vec{X}}{dt} = \vec{q}\left(X^i, t\right) \quad \text{.....................} \ (1.10)$$

(with initial conditions corresponding to $t = 0, \vec{X} = \vec{\xi}$) leads to the pathlines $\vec{X} = \vec{X}\left(\vec{\xi}, t\right)$. The differential equation (1.10) shows that the path of a point in the material is always tangential to its velocity. In this interpretation, the path line is the tangent curve to the velocities of the same material point at different times. Time is the curve parameter, and the material coordinate $\vec{\xi}$ is the family parameter.

A stream line or line of flow is a curve drawn in the fluid such that at any instant of time the tangent at any point of it is the direction of the motion of the fluid at the point.

stream lines are solutions of three simultaneous equations

$$\frac{dx^i}{ds} = q_i\left(x^1, x^2, x^3, t\right) \quad \text{.........................} \ (1.11)$$

where s is parameter along the stream lines. The parameter s is different from time occuring in the equation (1.10). The time t is taken fixed while integrating the equation (1.11), which gives the stream lines at the instant t. The stream lines alter from instant to instant.

From the definition of streamlines, it is clear that if two stream lines intersect each other then the fluid particle at the point of intersection will have two directions of velocity which is meaningless unless the velocity is taken to be zero at that point . Hence the point where two stream lines intersect has zero velocity, and such a point is called the stagnation point.

A streak line is a line on which lie all those fluid particles that at some earlier instant passed through a certain point in the fluid. Thus a streak line presents the instantaneous picture of the positions of all fluid particles, which have passed through a given point at some previous time. Examples of streak lines are smoke trails from chimneys or moving jets of water.

The equation of the streak line at time t can be derived by Lagrangian method. At time t the streak line through a fixed point $\vec{\alpha}$ is a curve going from $\vec{\alpha}$ to $\vec{x}$ ($\vec{\alpha}$, t). A particle is on the streak line if it passed the fixed point $\vec{\alpha}$ at some time between o and t. If this time were τ, then the material coordinate of the particle will be given by $\vec{\xi} = \vec{\xi}(\vec{\alpha}, \tau)$. However, at time t the particle is given by $\vec{x} = \vec{x}\left(\vec{\xi}, t\right)$. So that the equation of the streak line at time t is given by

$$\vec{X} = \vec{X}[\vec{\xi}(\vec{\alpha}, \tau), t)] \text{where } 0 \le \tau \le t.$$

To obtain the streak lines, first we solve the equation (1.10) to get the pathlines as

$$X^i = f^i(\xi^1, \xi^2, \xi^3, t) \qquad \text{... (1.12)}$$

and the X^i is replaced by α^i and t by τ to get

$$\alpha^i = f^i(\xi^1, \xi^2, \xi^3, \tau) \qquad \text{... (1.13)}$$

The curve obtained by eliminating ξ^1, ξ^2, ξ^3 between the equations (1.12) and (1.13) will give streaklines as

$$X^i = \phi^i(\alpha^1, \alpha^2, \alpha^3, t, \tau) \qquad \text{... (1.14)}$$

Pathlines, streamlines and streaklines coincide for steady motion.

1.7 Velocity Potential, rotational and irrotional motion

The equation (1.11) for stream lines can be written in cartesian co-ordinates as

$$\frac{dx}{u} = \frac{dz}{v} = \frac{dz}{w} \qquad \text{...(1.15)}$$

where u, v, w are velocity components in the directions of x,y,z respectively. The stream lines are cut perpendicularly by the surfaces given by the differential equation

$$udx + vdy + wdz = 0 \qquad \text{... (1.16)}$$

The condition that (1.16) is integrable is

$$\vec{q} \cdot \left(\text{curl}\, \vec{q} \right) = 0 \qquad \text{... (1.17)}$$

Since the fluid is in motion $\vec{q} \neq 0$, in general, at all points, hence if $\text{curl}\, \vec{q} = 0$ the equation (1.17) is always satisfied i.e. if the velocity vector $\vec{q}$ can be expressed as the gradient of a scalar function ϕ as

$$\vec{q} = -\vec{\nabla}\phi \qquad \text{... (1.18)}$$

This scalar function ϕ is known as velocity potential. The negative sign in the equation (1.18) is mere a convention and assures that the flow takes place from higher potential to lower potential. When (1.18) holds, the flow is said to be of the potential kind. It is also said to be irrotational for $\vec{q} = -\vec{\nabla}\phi \Rightarrow \text{curl}\,\vec{q} = 0$. The equation (1.17) may also be satisfied if $\vec{q}$ is perpendicular to $\text{curl}\,\vec{q}$ even though $\vec{q} \neq 0$. In such cases, the velocity potential will not exist inspite of the fact that surfaces cutting the stream lines orthogonally exist.

1.8 **Conservation of Mass and continuity equation** :

There are various approaches for deducing conservation laws. We adopt that one in which a certain volume of a fluid is isolated by an imaginary closed surface from the rest of the fluid and it is considered that the effect of the entire fluid on the isolated portion is taking place through the surface only. This Principle is called the principle of isolation. Let us apply this principle to deduce the equation of continuity.

Consider an arbitrary closed fixed surface S lying entirely in the fluid enclosing a volume V. Let $\vec{n}$ denote a unit vector in the direction of a normal drawn outwards at a point on a surface element ds of the surface S. Taking the velocity vector as $\vec{q}$, the rate at which the fluid flows into the surface through the boundary is given by

$$-\int_S \rho\left(\vec{q}\,.\,\vec{n}\right) ds \qquad \text{... (1.19)}$$

where ρ is the density of the fluid. The expression (1.19) can be written, by using Gauss's theorem, as

$$-\int_V \vec{\nabla}.\left(\rho\,\vec{q}\right) dv \qquad \text{... (1.20)}$$

The mass of the fluid within the volume V is

$$\int_V \rho dv \qquad \text{... (1.21)}$$

If no fluid is created or annihilated within the surface S then the mass can only increase by the flow through the boundary. So, by the law of conservation of mass, the rate of increase of the mass of fluid within V must be equal to the total rate of mass flowing into V. Hence, we have

$$\frac{\partial}{\partial t}\int_V (\rho dv) = -\int_V \vec{\nabla}.\left(\rho \vec{q}\right) dv$$

or $$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \vec{q}\right)\right] dv = 0 \qquad \text{... (1.22)}$$

Since the equation (1.22) holds good for every arbitrary volume V enclosed by an arbitrary surface S therefore the integrand itself must be zero i.e.

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\left(\rho \vec{q}\right) = 0 \qquad \text{... (1.23)}$$

which can also be written as -

$$\frac{\partial \rho}{\partial t} + \left(\vec{q}.\vec{\nabla}\rho\right) + \rho \vec{\nabla}.\vec{q}\, dv = 0$$

or $$\left(\frac{\partial}{\partial t} + + \vec{q}.\vec{\nabla}\right)\rho + \rho \vec{\nabla}.\vec{q} = 0$$

or $$\frac{d\rho}{dt} + \rho \left(\vec{\nabla}.\vec{q}\right) = 0 \qquad \text{... (1.24)}$$

The equation (1.24) is known as the equation of continuity. In the case when the fluid is incompresible $\frac{d\rho}{dt} = 0$ and since $\rho \neq 0$ hence we have

$$\vec{\nabla}.\vec{q} = 0 \qquad \text{... (1.25)}$$

The equation of continuity in orthogonal curvilinear coordinates can be written as

$$\frac{\partial \rho}{\partial t}+\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(\rho u h_2 h_3\right)+\frac{\partial}{\partial x^2}\left(\rho v h_3 h_1\right)+\frac{\partial}{\partial x^3}\left(\rho w h_1 h_2\right)\right]=0 \quad ...(1.26)$$

where u,v,w are velocity components along the direcctions of x_1,x_2,x_3 respectively. The equation of continuity in cartesian co-ordinate system (x,y,z) is given by

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0 \quad ...(1.27)$$

where u,v,w are velocity components in the directions of x,y,z respectively.

The values of h_1, h_2 and h_3 in cylindrical polar coordinates are 1, r and 1 respectively. Hence, by using the equation (1.26) the equation of continuity in cylindrical polar coordinates (r, θ, z) can be written as

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}.\frac{\partial}{\partial r}\left(\rho q_r r\right)+\frac{1}{r}.\frac{\partial}{\partial \theta}\left(\rho q_\theta\right)+\frac{\partial}{\partial z}\left(\rho q_z\right)=0 \quad ...(1.28)$$

where q_r, q_θ, q_z are vel. components in the direction of r, θ, z respectively.

In spherical polar co-ordinates h_1= 1, h_2= r and h_3= r sinθ and hence the equation of continuity in (r,θ,ϕ) system can be written as

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}.\frac{\partial}{\partial r}\left(\rho u r^2\right)+\frac{1}{r\sin\theta}.\frac{\partial}{\partial \theta}\left(\rho v \sin\theta\right)+\frac{1}{r\sin\theta}.\frac{\partial}{\partial \phi}\left(\rho w\right)=0 \quad ..(1.29)$$

where u, v, w are velocity components in the directions of r,θ,ϕ respectively

1.9 **Boundary Condition and Boundary Surface** :

When an inviscid fluid is in contact with a rigid solid surface or with another immisible fluid then the fluid and the surface (in contact) must have the same velocity normal to the surface. No condition is put on the tangential component of velocity of the fluid.

The pressure of the fluid must act in a direction normal to the boundary.

At the surface of separation of two immiseble fluids, the pressure must be comtinuous when we pass from on side of the surface to the other side.

A surface is said to be a boundary surface if it preserves contact with the fluid.

The contact between the fluid and the surface will be maintained if the fluid and the surface have the same velocity along the normal to the surface. Let P be a point on the moving boundary surface $F\left(\vec{r},t\right)=0$ or F(x,y,z,t)=0, where the fluid velocity is $\vec{q}$ and the velocity of the surface is $\vec{u}$, now since the normal component of the velocity of the fluid is equal to th normal component of the velocity of the surface, we have

$$\vec{q}.\vec{n}=\vec{u}.\vec{n} \qquad \text{... (1.30)}$$

when $\vec{n}$ is the unit normal vector drawn at the P on the boundary of the surface.

Since $\vec{\nabla}F$ is in the direction of the normal to the surface $F\left(\vec{r},t\right)=0$ hence $\vec{n}$ and $\vec{\nabla}F$ are paralled to each other. Thus (1.30) can be written in the form

$$\vec{q}.\vec{\nabla}F=\vec{u}.\vec{\nabla}F$$

Let $P\left(\vec{r},t\right)$ move to a point $Q\left(\vec{r}+\delta\vec{r},t+\delta t\right)$ in time δt.

Since Q also lies on the surface $F\left(\vec{r},t\right)=0$ hence $F\left(\vec{r}+\delta\vec{r},t+\delta t\right)=0$.

Expanding this equation by Taylor's theorem and retaining upto first order terms, we get

$$F\left(\vec{r},t\right)+\delta\vec{r}.\vec{\nabla}F+\delta t\left(\frac{\partial F}{\partial t}\right)=0$$

Also $F\left(\vec{r},t\right)=0$. Hence above equation becomes

$$\frac{\delta\vec{r}}{\delta t}.\vec{\nabla}F+\frac{\partial F}{\partial t}=0 \qquad \text{... (1.31)}$$

Taking the limit as δt tends to zero, equation (1.31) becomes

$$\frac{\partial F}{\partial t}+\frac{d\vec{r}}{dt}.\vec{\nabla}F=0$$

$$\Rightarrow\frac{\partial F}{\partial t}+\vec{u}.\vec{\nabla}F=0 \qquad \text{... (1.32)}$$

$$\Rightarrow\frac{\partial F}{\partial t}+\vec{q}.\vec{\nabla}F=0 \qquad (\text{since } \vec{q}.\vec{\nabla}F=\vec{u}.\vec{\nabla}F)$$

$$\Rightarrow\left(\frac{\partial}{\partial t}+\vec{q}.\vec{\nabla}\right)F=0$$

$$\Rightarrow\frac{dF}{dt}=0$$

This is the condition for the surface to be a possible form of boundary surface. The normal component of the velocity of the boundary is given by

$$\vec{u}.\vec{n}$$

$$=\vec{u}.\frac{\vec{\nabla}F}{\left|\vec{\nabla}F\right|}$$

$$=-\frac{\frac{\partial F}{\partial t}}{\left|\vec{\nabla}F\right|} \qquad \text{(by using 1.32)}$$

$$=-\frac{\frac{\partial F}{\partial t}}{\left|\vec{i}\frac{\partial F}{\partial x}+\vec{j}\frac{\partial F}{\partial y}+\vec{k}\frac{\partial F}{\partial z}\right|}$$

1.10 **Worked Example** :

Example 1. The components of velocity of a flow in cylindrical polar coordinates are $\left(r^2 z\cos\theta, r\,z\sin\theta, z^2 t\right)$. Find the components of the acceleration of a fluid particle.

Solution : Let u, v, w denote the velocity components in cylindrical polar coordinates. It is given that

$$u = r^2 z\cos\theta, \quad v = r\,z\sin\theta, \quad w = z^2 t \qquad \ldots (A)$$

If f_r, f_θ, f_z denotes the components of acceleration in cylindrical polar coordinates then

$$f_r = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + \frac{v}{r}\frac{\partial u}{\partial z} - \frac{v^2}{r}$$

$$= 0 + \left(r^2 z\cos\theta\right)\left(2r\,z\cos\theta\right) + \left(\frac{r\,z\sin\theta}{r}\right)\left(-r^2 z\sin\theta\right) + \left(z^2 t\right)\left(r^2\cos\theta\right) - \frac{\left(rz\sin\theta\right)^2}{r}$$

$$f_r = rz^2\left(2r^2\cos^2\theta - r\sin^2\theta + r\,t\cos\theta - \sin^2\theta\right)$$

In a like manner f_θ and f_z can be calculated.

Example 2 : If the velocity components u,v,w, in cylindrical polar co-ordinates are

$$u = -U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \quad v = U\left(1 + \frac{a^2}{r^2}\right)\sin\theta, \quad w = 0$$

then derive the equation of stream lines.

Solution : The equation of stream line is $\vec{q} \times d\vec{r} = 0$ which in orthogonal curvilinear coordinate system can be written as

$$\frac{h_1 dx_1}{u} = \frac{h_2 dx_2}{v} = \frac{h_3 dx_3}{w} \qquad \ldots (A)$$

In cylindrical polar coordinates system

$h_1=1, \qquad h_2 = r, \qquad h_3 = 1$

and $\quad dx_1 = dr \qquad dr_2 = d\theta, \qquad dr_3 = dz$

Hence equation (A), in this case, becomes

$$\frac{dr}{u} = \frac{rd\theta}{v} = \frac{dz}{w}$$

Now putting the given values of u,v, w in above equation, we get,

$$\frac{dr}{-U\left(1-\frac{a^2}{r^2}\right)\cos\theta} = \frac{rd\theta}{U\left(1+\frac{a^2}{r^2}\right)\sin\theta} = \frac{dz}{0}$$

$$\Rightarrow dz = 0, \frac{U\left(1+\frac{a^2}{r^2}\right)dr}{-Ur\left(1-\frac{a^2}{r^2}\right)} = \frac{\cos\theta}{\sin\theta}d\theta$$

$$\Rightarrow z = \text{constant}, \frac{r^2+a^2}{r\left(r^2-a^2\right)}dr = -\cot\theta\, d\theta$$

$$\Rightarrow z = c_1, \frac{2rdr}{r^2-a^2} - \frac{dr}{r} = -\cot\theta\, d\theta$$

$$\Rightarrow z = c_1, \log\frac{r^2-a^2}{r} = -\log\sin\theta + \log C_2$$

$$\Rightarrow z = c_1, \left(\frac{r^2-a^2}{r}\right)\sin\theta = C_2$$

$$\Rightarrow z = c_1, \left(r-\frac{a^2}{r}\right)\sin\theta = C_2$$

These are the required equations of the the stream lines.

Example 3 : Does a velocity field given by

$$\vec{q} = 5x^3\,\vec{i} - 15x^2y\,\vec{j} + t\,\vec{k}$$

represent a possible incompressible flow of fluid?

Solution : In order to check for a physically possible incompressible fluid flow, on has to look for its compliance with the equation of continuity.

The equation of continuity, for a three dimensional incompressible flow, in cartesian coordinate system can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Here, $u = 5x^3$, $v = -15x^2y$ and $w = t$

Hence, $\frac{\partial u}{\partial x}=15x^2$, $\frac{\partial v}{\partial y}=-15x^2$ and $\frac{\partial w}{\partial z}=0$

which, on substitution in the continuity equation satisfies it for all values of x,y,z and t values. This shows that the above velocity field represents a physically possible incompressible flow.

Example 4 : show that the surface $\left(x^2/a^2\right)\tan^2 t+\left(y^2/b^2\right)\cot^2 t=1$ is a possible form for the bounding surface of a liquid, and find an expression for the normal component of velocity.

Solution : Here, $F(x, y, t)=\frac{x^2}{a^2}\tan^2 t+\frac{y^2}{b^2}\cot^2 t-1=0$

F(x,y,t) can be a possible bounday surface, if it satisfies the boundary condition.

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}=0 \qquad \text{... (A)}$$

Here, $\frac{\partial F}{\partial t}=\frac{x^2}{a^2}2.\tan t.\sec^2 t-\frac{y^2}{b^2}2.\cot t.\text{cosec}^2 t$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2}\tan^2 t,\ \frac{\partial F}{\partial y}=\frac{y^2}{b^2}\cot^2 t$$

with these values, (A) reduces to

$$\frac{2x\tan t}{a^2}\left(x\sec^2 t+u\tan t\right)+\frac{2y\cot t}{b^2}\left(-y\,\text{cosec}^2 t+v\cot t\right)=0$$

which will be identically satisfied if we take

$$x \sec^2 t + u \tan t = 0 \text{ and } -y \operatorname{cosec}^2 t + v \cot t = 0$$

i.e. $$u = \frac{x}{\sin t . \cos t} \quad \text{and} \quad v = \frac{y}{\sin t . \cos t}$$

i.e. $$\frac{\partial u}{\partial x} = \frac{-1}{\sin t . \cos t} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{1}{\sin t . \cos t}$$

These values of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ satisfy the equation of continuity for a liquid viz. $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. Hence these values of u and v represent a possible motion of a liquid and therefore the given surface represent a possible boundary surface.

Normal component of velocity

$$= \frac{u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y}}{\sqrt{\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2}}$$

$$= \frac{-\frac{x}{\sin t . \cos t} . \frac{2x \tan^2 t}{a^2} + \frac{y}{\sin t . \cos t} . \frac{2y \cot^2 t}{b^2}}{\sqrt{\left(\frac{2x \tan^2 t}{a^2}\right)^2 + \left(\frac{2y \cot^2 t}{b^2}\right)^2}}$$

$$= \frac{a^2 y^2 \cot t . \operatorname{cosec}^2 t - b^2 x^2 \tan t . \sec^2 t}{\sqrt{x^2 b^4 \tan^4 t + y^2 b^4 \cot^4 t}}$$

1.11 **Check your progress** :

1. Does a velocity field given by

$$\vec{q} = \frac{k^2}{x^2 + y^2}\left(x \vec{i} - y \vec{j}\right) \qquad (k = \text{constant})$$

represent a possible incompressible flow of fluid ? If so, determine the equations of the stream lines. Also test whether the motion is of the potential kind and if so determine the velocity potential.

2. If the velocity components u,v,w in spherical polar co-ordinates are

$$u = U\left(1-\frac{a^3}{r^3}\right)\cos\theta,\ v = -U\left(1+\frac{a^3}{2r^3}\right)\sin\theta,\ \ w = 0$$

then derive the equations of the stream lines.

3. Steam is rushing from a boiler throught a conical pipe, the diameters of the ends of which are D and d; if V and v be the corresponding velocities of the stream, and if the motion is supposed to be that of divergence from the vertex of the cone, prove that

$$\frac{v}{V} = \frac{D^2}{d^2}\ \exp\ \left(\frac{v^2 - V^2}{2k}\right)$$

where k is the pressure divided by the density, and supposed constant.

4. Liquid is contained between two parallel planes; the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated. Prove tha if $\overset{\square}{\omega}$ be the pressure at outer surface, the initial pressure at any point of the liquid distant r from the centre is

$$\overset{\square}{\omega} = \frac{\log r\ -\ \log b}{\log a\ -\ \log b}$$

5. Explain the method of differntiation following the fluid, and find the condition that the surface F(x, y, z, t) = 0 may be a boundary surface. Prove that the variable ellipsoid

$$\frac{x^2}{a^2k^2t^4} + Kt^2\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 1$$

is a possible form of boundary surface.

6. An infinite mass of fluid is acted on by a force $\mu/r^{3/2}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere r = c in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{1/2} C^{5/4}$.

7. The velocity components in a three dimensional flow field for an incompressible fluid are (2x,-y,-z). Is it a possible field? Determine the equations of the stream line passing through the point (1,1,1).

1.11 **Let us sum up** :

Points to Remember :

- The basic mathematical idea of fluid motion is that it can be described by a point transformation.
- All the fluid properties (pressure, velocity, density, tempreature, etc.) of a system, in continuum approach, are functions of space coordinates and time
- A fluid is a substance that deforms continuously when subjected to even an infinitesimal shear stress.
- Kinematics of fluid deals with the geometry of fluid motion. It characterizes the different types of motion.
- A flow is defined to be steady when the hydrodynamic parameters and fluid properties at any point do not change with time.
- A stream line at any instant of time is an imaginary curve or line in the flow field so that the tangent to

the curve at any point represents the direction of the instantaneous velocity at that point. A path line is the trajectory of a fluid particle of a given identity. A streak line at any instant of time is the locus of temporary locations of all particles that have passed through a fixed point in the flow. In a steady flow, the streamline, path lines and streak lines are identical.

- The existance of a physically possible flow field is verified from the principle of a conservation of mass. Continuity equation is the equation of conservation of mass in a fluid flow. The general form of the continuity equation for an unsteady compressible flow is given by

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} . \left(\rho \vec{q} \right) = 0$$

where $\vec{q}$ is the velocity vector.

- A fluid element in motion possesses intermolecular energy, kinetic energy and potential energy.

1.13 **Further readings** :

1. Fluid Mechanics, Landau and Lifshitz. Butterworth-Heinemann, Oxford.

2. Theoretical Hydrodynamics by L.M. Milne-Thomson.

BLOCK -2 : THE EQUATION OF MOTION OF AN INVISCID FLUID

UNIT – I : THE EQUATION OF MOTION OF AN INVISCID FLUID

Structure

2.0 **Objective** :

In this unit the governing equation of motion of fluid will be derived. the governing equation will be derived for ideal fluid where we consider fluid motion with the no viscous effect. The force in action in the motion will be pressure force. The gravitational force are often neglected. The equation of motion thus obtained will be on consideration for conservation of momentum of fluid element.

If the forces in action are conservative and flow is of potential kind, then the equation will take special form which will also be discussed here.

The equation of motion for the forces which act only for a very short duration will also be discussed in this unit.

2.1 **Introduction** :

Historically, there have been two different approaches taken to describe the equation of fluid dynamics : the phenomenological approach and kinetic theory approach. In phenomenological approach, certain relation between stress and rate of strain are postulated and the fluid dynamics equation are then developed from the conservation laws. In the kinetic theory approach (also called Mathematical theory of non uniform gases) the fluid dynamic equations are obtained with transport coefficients defined in terms of certain internal relations. In this approach mathematical uncertainity takes place contrary to the involvement of experimental uncertainty in phenomenological approach. Here our approach will be Mathematical.

The governing equation of motion that will be presented will be derived from conservative laws of momentum which is nothing more than Newton's second law of motion. When this law is applied to an element of fluid flow, it yields a vector equation known as momentum equation. In the context of inviscid fluid we call it Euler's equation of motion.

2.2 **Euler's equation of Motion :**

Consider a non-viscous fluid occupying a certain region. In this region let V be the volume enclosed by a surface S which moves with the fluid and so contains the same fluid particles during its motion. Within the surface S let dv be the volume element surrounding a fluid particle P of density ρ, since the volume element is moving with the fluid hence the mass $\rho\,dv$ of this fluid element remains constant all times. Let $\vec{q}$ denote the velocity of the fluid particle P and ρ denote the pressure at a point of the surface element ds which has an outward drawn normal $\vec{n}$.

Newton's second law of motion states that the rate of change of linear momentum is equal to the sum of forces.

The forces are due to (i) the pressure which is acting at each point of the boundary in the normal directions; (ii) the external force $\vec{F}$ per unit mass acting on the fluid within the surface S.

Thus the total forfce acting on the volume V is

$$\int_S P\vec{n}\,ds + \int_V \rho\vec{F}\,dV = -\int_V \left(\vec{\nabla} P\right) dV + \int_V \rho\vec{F}\,dv$$

using Gauss's theorem. Equating this force to the rate of change of linear momentum $\frac{d}{dt}\int_V \rho dv\,\vec{q}$ we get

$$\frac{d}{dt}\int_V \rho dv\,\vec{q} = -\int_V \left(\vec{\nabla} p\right) dv + \int_V \rho\vec{F}\,dv$$

or
$$\int_V \left(\frac{d\vec{q}}{dt}\right)\rho dv + \int_V \vec{q}\,\frac{d}{dt}(\rho dv) = \int_V \left(\rho\vec{F} - \vec{\nabla} p\right) dv$$

Noting that $\frac{d}{dt}(\rho dv) = 0$, above equation becomes

$$\int_V \left[\rho \frac{d\vec{q}}{dt} - \left(\rho \vec{F} - \vec{\nabla} p \right) \right] dv = 0$$

Since the volume of integration is arbitrary hence above equation will be identically satisfied if we take integrand equal to zero, i.e.

$$\rho \frac{d\vec{q}}{dt} - \left(\rho \vec{F} - \vec{\nabla} p \right) = 0$$

or $$\rho \frac{d\vec{q}}{dt} = \rho \vec{F} - \vec{\nabla} p \qquad \text{... (2.1)}$$

This equation is known as Euler's equation of motion.

Above equation can also be writen as

$$\frac{\partial \vec{q}}{\partial t} + \left(\vec{q} . \nabla \right) \vec{q} = \vec{F} - \frac{1}{\rho} \vec{\nabla} P \qquad \text{... (2.2)}$$

or $$\frac{\partial q}{\partial t} + \vec{\nabla} \left(\frac{1}{2} . q^2 \right) - \vec{q} \text{ x curl } \vec{q} = \vec{F} - \frac{1}{\rho} \vec{\nabla} P \qquad \text{... (2.3)}$$

In the case when the body force $\vec{F}$ is conservative and that the flow is of the potential kind, there exist scalar functions Ω and ϕ such that

$$\vec{F} = -\vec{\nabla} \Omega \qquad \text{and} \qquad \vec{q} = -\vec{\nabla} \phi$$

In such a case curl $\vec{q} = 0$ and equation (2.3) becomes

$$-\vec{\nabla} \left(\frac{\partial \phi}{\partial t} \right) + \vec{\nabla} \left(\frac{1}{2} . q^2 \right) = -\vec{\nabla} \Omega - \frac{1}{\rho} \vec{\nabla} p \qquad \text{... (2.4)}$$

Taking dot product of (2.4) by $d\vec{r}$ and noting that

$\vec{\nabla} f . d\vec{r} = df$, We get

$$d\left(-\frac{\partial \phi}{\partial t} \right) + d\left(\frac{1}{2} . q^2 \right) = -d\Omega - \frac{1}{\rho} dp$$

Integrating and rearranging the terms of the above equation we get

$$\frac{q^2}{2}+\Omega-\frac{\partial \phi}{\partial t}+\int \frac{dp}{\rho}=f(t) \qquad \text{... (2.5)}$$

where $f(t)$ is an arbitrary function of t. This equation is known as Bernoulli's equation in its most general form. In the case of steady flow (2.5) takes the form

$$\frac{q^2}{2}+\Omega+\int \frac{dp}{\rho}=\text{Constant} \qquad \text{... (2.6)}$$

Further, if the fluid is homogeneous and in-compressible so, that ρ is constant, (2.6) becomes

$$\frac{q^2}{2}+\Omega+\frac{p}{\rho}=\text{Constant} \qquad \text{... (2.7)}$$

In absence of external body forces (2.7) reduces to

$$p+\tfrac{1}{2}\rho q^2=\text{Constant}$$

2.3 **Impulsive Motion** :

If sudden velocity changes occur at the boundaries of an incompressible fluid or impulsive forces are made to act its interior then the impulsive pressure at any point is the same in every direction and the distrubances produced in both cases are propagated instantaneously throughout the fluid.

consider an arbitrary closed surface S moving with a non-viscous fluid such that it encloses a volume V. Let us suppose that the fluid is subjected to external impulses $\vec{I}$ per unit mass and to impulsive pressure ω on a surface element ds. Also let $\vec{n}$ denote the unit outward normal vector. If $\vec{q}_2$ is the velocity generated in the element which was previously moving with velocity $\vec{q}_1$ then by applying Newton's second law of motion, we get,

Total body forces = Change of momentum

$$\Rightarrow \int_V \vec{I}\rho dV + \int_S -\vec{n}\,\omega ds = \int_V \rho\left(\vec{q}_2 - \vec{q}_1\right) dv$$

where ρ is the density in the fluid element.

Using Gauss's theorem we get

$$\int_V \vec{I}\rho dV - \int_V \vec{\nabla}\omega dv = \int_V \rho\left(\vec{q}_2 - \vec{q}_1\right) dv$$

$$\Rightarrow \int_V \left\{\vec{I}\rho - \vec{\nabla}\omega - \rho\left(\vec{q}_2 - \vec{q}_1\right)\right\} dv = 0$$

Since the volume of integration is arbitrary hence the integrand of the last integral vanishes.

$$\therefore \vec{I}\rho - \vec{\nabla}\omega - \rho\left(\vec{q}_2 - \vec{q}_1\right) = 0$$

or
$$\vec{I} - \frac{1}{\rho}\vec{\nabla}\omega = \vec{q}_2 - \vec{q}_1 \qquad \text{... (2.8)}$$

This is the general equation of impulsive motion. In the special case when external impulsive body forces are absent, i.e. $\vec{I} = 0$ whereas impulsive pressures are present then equation (2.8) reduces to

$$-\frac{1}{\rho}\vec{\nabla}\omega = \vec{q}_2 - \vec{q}_1 \qquad \text{... (2.9)}$$

$$\Rightarrow \vec{\nabla}.\left[-\frac{1}{\rho}\vec{\nabla}\omega\right] = \vec{\nabla}.\left[\vec{q}_2 - \vec{q}_1\right] \qquad \text{... (2.10)}$$

Further, if the fluid is incompressible, the equation of continuity gives

$$\vec{\nabla}.\vec{q}_1 = 0 = \vec{\nabla}.\vec{q}_2 \qquad \text{... (2.11)}$$

Using (2.11) in (2.10), we get

$$\nabla^2 \varpi = 0 \qquad \text{... (2.12)}$$

Above equation states that in absence of external body forces, for an incompressible fluid, impulsive pressure satisfies Laplace's equation.

In the case when external impulsive body forces are absent, i.e. $\vec{I} = 0$ and the flow is of potential kind (i.e. $\vec{q}_1 = -\vec{\nabla}\phi$) then

$$\vec{q}_1 = -\vec{\nabla}\phi_1 \text{ and } \vec{q}_2 = -\vec{\nabla}\phi_2 \qquad \text{... (2.13)}$$

Therefore equation (1.40) reduces to

$$-\frac{1}{\rho}\vec{\nabla}\varpi = -\vec{\nabla}\phi_2 + \vec{\nabla}\phi_1$$

$$\Rightarrow \vec{\nabla}\varpi = \rho\left(\vec{\nabla}\phi_2 - \vec{\nabla}\phi_1\right)$$

$$\Rightarrow \left(\vec{\nabla}\varpi\right).d\vec{r} = \rho\left(\vec{\nabla}\phi_2 - \vec{\nabla}\phi_1\right).d\vec{r}$$

$$\Rightarrow d\varpi = \rho\left(\phi_2 - \phi_1\right)$$

Integrating when ρ is constant, get,

$$\Rightarrow \varpi = \rho\left(\vec{\nabla}\phi_2 - \vec{\nabla}\phi_1\right) + C \qquad \text{... (2.14)}$$

The constant C may be ommited as an extra pressure, constant throughout the fluid, would not effect the motion.

From above analysis it can be concluded that a potential motion can be produced instantaneously from rest by a set of impulses properly applied at any point and the velocity potential, in this case, is the impulsive pressure divided by the density.

2.4 The Rate of Change of Energy :

Theorem : The rate of change of total energy (Kinetic, potential and intrinsic) of any portion of a fluid as it moves about is equal to the rate of working of the pressure on the boundary if the potential due to the external forces are independent of time.

Proof : Consider an arbitrary closed surface S drawn in the fluid let V be the volume of the fluid within S. Let ρ be the density of the fluid particle at the point P and dv be the volume element surrounding the point P. Let $\vec{q}\left(\vec{r},t\right)$ be the velocity of the fluid particle.The Euler's equation of motion is

$$\rho\frac{d\vec{q}}{dt}=-\vec{\nabla}p+\rho\vec{F} \qquad \text{... (2.15)}$$

Taking the external forces to be conservative i.e. $\vec{F}=-\nabla\Omega$, considering Ω to be independent of time i.e. $\frac{\partial\Omega}{\partial t}=0$ and taking the scalar product of (2.15) with $\vec{q}$, we get

$$\rho\vec{q}.\frac{d\vec{q}}{dt}=-\vec{q}.\vec{\nabla}p-\rho\vec{q}.\vec{\nabla}\Omega$$

or $$\rho\left[\frac{d}{dt}\left(\frac{1}{2}q^2\right)+\left(\vec{q}.\vec{\nabla}\right)\Omega\right]=-\vec{q}.\vec{\nabla}p$$

or $$\rho\left[\frac{d}{dt}\left(\frac{1}{2}q^2\right)+\frac{\partial\Omega}{\partial t}+\left(\vec{q}.\vec{\nabla}\right)\Omega\right]=-\vec{q}.\vec{\nabla}p$$

or $$\rho\left[\frac{d}{dt}\left(\frac{1}{2}q^2\right)+\frac{d}{dt}\Omega\right]=-\vec{q}.\vec{\nabla}p$$

or $$\rho\frac{d}{dt}\left(\frac{1}{2}q^2+\Omega\right)=-\vec{q}.\vec{\nabla}p$$

Multiplying the above equation by the volume element dV and integrating over the volume V, we get,

$$\int_V \rho \frac{d}{dt}\left(\frac{1}{2}q^2 + \Omega\right) dV = -\int_V \left(\vec{q}.\vec{\nabla} p\right) dV$$

$$\Rightarrow \frac{d}{dt}\left[\int_V \rho\left(\frac{1}{2}q^2 + \Omega\right) dV\right] = -\int_V \left(\vec{q}.\vec{\nabla} p\right) dV$$

$$\left(\because \frac{d}{dt}(\rho dV) = 0\right)$$

$$\Rightarrow \frac{d}{dt}\left[\int_V \frac{1}{2} pq^2 dV + \int_V \rho\Omega dV\right] = -\int_V \vec{\nabla}.\left(p\vec{q}\right) dV + \int_V p\nabla.\vec{q}\, dV \qquad \text{... (2.16)}$$

The kinetic energy T, potential energy W and internal energy I of the fluid are given by

$$T = \int_V \frac{1}{2}\rho q^2 dV, \qquad W = \int_V \rho\Omega dV, \qquad I = \int_V \rho E dV \qquad \text{... (2.17)}$$

where E is the internal energy per unit mass.

It can be shown that

$$\rho\frac{dE}{dt} = -p\left(\nabla.\vec{q}\right)$$

Hence

$$\frac{dI}{dt} = \frac{d}{dt}\int_V \rho E dV = \int_V \rho\frac{dE}{dt} dV = -\int_V p\vec{\nabla}.\vec{q}\, dV \qquad \text{... (2.18)}$$

Using (2.17) and (2.18) in (2.16), we get,

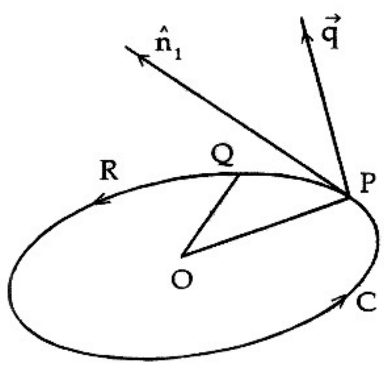
$$\frac{d}{dt}[T+W+I] = \int_V \nabla.\left(p\vec{q}\right) dV$$

$$\Rightarrow \frac{d}{dt}[T+W+I] = \int_S \vec{n}.\left(p\vec{q}\right) ds \qquad \text{... (2.19)}$$

= the rate of working of pressure on the boundary.

2.5 Circulation and Kelvins Circulation Theorem :

Consider a closed curve C situated entirely in a moving fluid. Let $\vec{q}$ be the velocity vector and $\hat{n}$ be a vector drawn in the direction of tangent at an arbitrary point P of the curve. The direction of the tangent is so chosen that an observer moving from the point P in the sense of $\hat{n}$ describes the curve in the positive direction. Let Q be such a point on the curve, adjacent to P, that the arc PQ be of infinitesimal length δs . We form the scalar product $\vec{q}.\hat{n}.\delta s = \vec{q}.d\vec{s}$ at the point P, where $\delta \vec{s}$ is the directed element of the arc at the point.



Forming similar products at points Q, R, . . . and so on right round the curve back again to the point P, we define the circulation of the velocity vector round the curve C by the relation

$$\text{Circulation} = \underset{\delta s \to 0}{Lt} \sum \vec{q}.d\vec{s}$$

$$= \int_C \vec{q}.d\vec{s}$$

$$\text{Circulation} = \int_C \vec{q}.d\vec{r} \qquad \ldots (2.20)$$

We can form the circulatuion of any vector round a curve in a like manner.

If the motion is of potential kind or (irrotational) then $\vec{q} = -\vec{\nabla}\phi$ where ϕ is the velocity potential. In this case (2.20) becomes

$$\text{Circulation} = -\int_C \vec{\nabla}\phi . d\vec{r}$$

$$= -\int_C d\phi$$

$$= [\phi]_C$$

It follows that if the motion is irotational and the velocity potential is single valued, the circulation in every closed circuit in the fluid is zero. However, if the region in which the irrotational motion takes place is not simply connected (A region in which every circuit can be shrunk to a point of the region without passing outside the region is known as simply connected), the circulation in any two reconciable circuits must be the same, but the velocity potential, will not, in general, be single valued.

Kelvin's Circulation Theorem :

Theorem : If the external forces are conservative and are derivable from a single valued potential function, and the density is a function of pressure only then the circulation in any closed circuit moving with the fluid is constant for all time.

Proof : Let C be a closed circuit moving with the fluid. Let $\vec{q}$ be the velocity at any point $P\left(\vec{r}\right)$ of the circuit. If Γ denote the circulation along the closed circuit C then from definition of circulation we get,

$$\Gamma = \int_C \vec{q} . d\vec{r}$$

$$\Rightarrow \frac{d}{dt}(\Gamma) = \frac{d}{dt}\int_C \vec{q} . d\vec{r}$$

$$\Rightarrow \frac{d\Gamma}{dt} = \int_C \frac{d}{dt}\left(\vec{q}.d\vec{r}\right)$$

$$\Rightarrow \frac{d\Gamma}{dt} = \int_C \left[\frac{d\vec{q}}{dt}.d\vec{r} + \vec{q}.\frac{d}{dt}.\left(d\vec{r}\right)\right]$$

$$\Rightarrow \frac{d\Gamma}{dt} = \int_C \left[\frac{d\vec{q}}{dt}.d\vec{r} + \vec{q}.d\vec{q}\right]$$

$$\Rightarrow \frac{d\Gamma}{dt} = \int_C \left[\left(F - \frac{1}{\rho}.\vec{\nabla} p\right).d\vec{r} + d\left(\frac{1}{2}q^2\right)\right]$$

(Using Euler's equation of motion)

$$\Rightarrow \frac{d\Gamma}{dt} = \int_C \left[\left(-\vec{\nabla}\Omega - \frac{1}{\rho}.\vec{\nabla} p\right).d\vec{r} + d\left(\frac{1}{2}q^2\right)\right]$$

$$\Rightarrow \frac{d\Gamma}{dt} = \int_C \left[-d\Omega - \frac{1}{\rho}dp + d\left(\frac{1}{2}q^2\right)\right]$$

$$\Rightarrow \frac{d\Gamma}{dt} = \int_C \left[-\Omega - \int\frac{dp}{\rho} + \frac{1}{2}q^2\right] \qquad \text{... (2.21)}$$

Since Ω, p and q are single valued functions of $\vec{r}$ therefore, on passing once round the circuit, the change expressed in (2.21) is zero.

thus

$$\frac{d}{dt}\Gamma = 0$$

which implies that the circulation is constant along C for all times.

2.6 Worked examples :

Example 1 : If a bomb shell explodes at a great depth beneath the surface of the sea, prove that the impulsive pressure at any point varies inversely as the distance from the centre of the shell.

Solution : In the present case if we use spherical polar coordinates then the velocity will be in the direction of r only, so the equation of continuity will be satisfied if $ur^2 = C$ where u is the velocity, C is a constant and origin is take as the point of explosion. there is no external impulse i.e. $\vec{I} = 0$ and the equation (2.9) gives

$$-\frac{1}{\rho}\frac{d\tilde{\omega}}{dr} = u = c/r^2$$

or $$\tilde{\omega} = \frac{C\rho}{r} + D$$

When $r \to \alpha$, $\tilde{\omega} \to 0$ gives D=0, we have

$$\tilde{\omega} = \frac{c\rho}{r}$$

which shows that the impulsive pressure $\tilde{\omega}$ is inversely proportional to the distance.

Example : Show that the velocity field

$$u(x,y) = \frac{B(x^2-y^2)}{(x^2+y^2)^2}, \quad v(x,y) = \frac{2Bxy}{(x^2+y^2)^2}, \quad w = 0$$

satisfies the equation of motion for an inviscid incompressible flow. Determine the pressure associated with the velocity field. Here B is constant.

Solution : Euler's equation of motion eg. (2.1) in the absence of external forces is

$$\frac{d\vec{q}}{dt} = -\frac{1}{\rho}\nabla p$$

or $$\left(\frac{\partial}{\partial t} + \vec{q}.\nabla\right)\vec{q} = -\frac{1}{\rho}\nabla p$$

But the motion is two dimentional w=0, $\vec{q} = ui + vj$

$$\therefore \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\vec{q} = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

Putting the value

$$\left[\frac{\partial}{\partial t}+\frac{B\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}\frac{\partial}{\partial x}+\frac{2Bxy}{\left(x^2+y^2\right)^2}\frac{\partial}{\partial y}\right](ui+vj)=-\frac{1}{\rho}\left(i\frac{\partial p}{\partial x}+j\frac{\partial p}{\partial y}\right)$$

As u,v are independent of t, by assumption

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}=0$$

Hence the last euqation gives

$$\frac{B}{\left(x^2+y^2\right)^2}\left[\left(x^2-y^2\right)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y}\right](ui+vj)=-\frac{1}{\rho}\left(i\frac{\partial p}{\partial x}+j\frac{\partial p}{\partial y}\right)$$

Equating the coefficients of i and j from both sides, we get

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{B}{\left(x^2+y^2\right)^2}\left[\left(x^2-y^2\right)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y}\right]\frac{B\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2} \quad \text{.........(1)}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{B}{\left(x^2+y^2\right)^2}\left[\left(x^2-y^2\right)\frac{\partial}{\partial x}+2xy\frac{\partial}{\partial y}\right]\frac{2Bxy}{\left(x^2+y^2\right)^2} \quad \text{.........(2)}$$

But $$\frac{\partial}{\partial x}\left\{\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\right\}=\frac{2x\left(3y^2-x^2\right)}{\left(x^2+y^2\right)^3} \quad \text{.........(3)}$$

$$\frac{\partial}{\partial y}\left\{\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\right\}=\frac{2y\left(3x^2-y^2\right)}{\left(x^2+y^2\right)^3} \quad \text{.........(4)}$$

$$\frac{\partial}{\partial x}\left\{\frac{2xy}{\left(x^2+y^2\right)^2}\right\}=\frac{2y\left(y^2-x^2\right)}{\left(x^2+y^2\right)^3} \quad \text{.........(5)}$$

$$\frac{\partial}{\partial y}\left\{\frac{2xy}{\left(x^2+y^2\right)^2}\right\}=\frac{2x\left(x^2-y^2\right)}{\left(x^2+y^2\right)^3} \quad \text{.........(6)}$$

we rewrite (1) with the help of (3) and (4) to get

$$\frac{\partial p}{\partial x}=\frac{-2\rho B^2}{\left(x^2+y^2\right)^5}\left[\left(x^2-y^2\right)x\left(3y^2-x^2\right)-2xy^2\left(3x^2-y^2\right)\right]$$

$$\frac{\partial p}{\partial x}=\frac{-2\rho B^2 x}{\left(x^2+y^2\right)^3} \qquad ....(7)$$

we rewrite (2) with the help of (5) and (6)

$$\frac{\partial p}{\partial y}=\frac{-2\rho B^2}{\left(x^2+y^2\right)^5}\left[\left(x^2-y^2\right)y\left(y^2-x^2\right)+2x^2y\left(x^2-y^2\right)\right]$$

$$\frac{\partial p}{\partial y}=\frac{2By\rho\left(x^2-y^2\right)}{\left(x^2+y^2\right)^4} \qquad ....(8)$$

Differentiation (7) and (8) with respect to y and x we find

$$\frac{\partial^2 p}{\partial y\partial x}=\frac{\partial^2 p}{\partial x\partial y}$$

This shows that velocity field satisfies the equation of motion

$$dp=\frac{\partial p}{\partial y}dx+\frac{\partial p}{\partial x}dy$$

Now from (7) and (8) we have

$$dp=2\rho B^2\left[\frac{xdx}{\left(x^2+y^2\right)^3}-\frac{y\left(x^2-y^2\right)}{\left(x^2+y^2\right)^4}dy\right]$$

$$=2\rho B^2\left[Mdx+Ndy\right] \qquad ....(9)$$

$$\frac{\partial M}{\partial y}=-\frac{2xy}{\left(x^2+y^2\right)^4}=\frac{\partial N}{\partial x}$$

$\therefore Mdx+Ndy$ is extract

$$\int Mdx+Ndy=\int\frac{xdx}{\left(x^2+y^2\right)^3}+\int 0dy$$

$$=\frac{1}{2}\int 2x\left(x^2+y^2\right)^{-3}dx+C$$

$$=-\frac{1}{4\left(x^2+y^2\right)^3}+C$$

In view of this (9) becomes

$$p = -\frac{2\rho B^2}{4\left(x^2+y^2\right)^2} + C_1$$

This is the required expression for pressure.

Example 3 : A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being π. Show that, if the radius R of the sphere varies in any manner the pressure at the surface of the sphere at any time is

$$\pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

If R =q(2+cosnt), show that, to prevent cavitation in the fluid, π must not be less than $3\rho a^2 n^2$

Working rules

In order to solve equation of motion we adopt the following techniques

1) Equation of motion is -

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = F - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{where } F = -\frac{\partial \Omega}{\partial x}, \text{ when}$$

velocity has one component

2) Equation of continuity -

(i) $x^2 v = F(t)$ for spherical symmetry
if ρ = Constant

(ii) $xv = F(t)$ for cylindrical symmetry
if ρ = Constant

(iii) $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$ general.

3) Fluid is assumed to be at rest at infinity -

$x = \propto$, $v = 0$, $p = \pi$

4) If r be the radius of cavity(or hollow sphere), then

$x = r$, $v = \hat{r}$, $p = \pi$

5) Equationof the impu;sive action is $d\tilde{w} = \rho v dx = \rho v' d\gamma'$.

Solution :

The equation of motion is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

and equation of continuity is $x^2u=F(t)$

So that $\frac{\partial u}{\partial t}=\frac{F^{/}(t)}{x^2}$

Hence $\frac{F^{/}(t)}{x^2}+\frac{\partial}{\partial t}\left(\frac{1}{\rho}u^2\right)=-\frac{\partial}{\partial t}\left(\frac{p}{\rho}\right)$ as ρ is constant.

Integrating w.r.t x $\frac{-F^{/}(t)}{x^2}+\frac{1}{\rho}u^2=-\frac{p}{\rho}+C$ ... (1)

Boundary conditions are -

(i) When $x=\infty \quad p=\pi \quad u=0$ ... (2)

(ii) When $x=R \quad p=p \quad u=\dot{R}$ ... (3)

Also $x^2u=F(t)=R^2\ddot{R}$

$$\therefore F^{/}(t)=2R\left(\dot{R}\right)^2+R^2R$$

Using (2) and (3) in (1)

$$0+0=-\frac{\pi}{\rho}+C \text{ and}$$

$$-\frac{F^{/}(t)}{R}+\frac{1}{2}\left(\dot{R}\right)^2=-\frac{p}{\rho}+C=-\frac{p}{\rho}+\frac{\pi}{\rho}$$

or $$\frac{p}{\rho}=\frac{\pi}{\rho}-\frac{1}{2}\left(\dot{R}\right)^2+\frac{1}{R}\left[2R\left(\dot{R}\right)^2+R^2R\right]$$

or $$p=\pi+\frac{1}{2}\rho\left[3\left(\dot{R}\right)^2+2R\ddot{R}\right] \quad \text{... (4)}$$

Now, $\frac{d^2R^2}{dt^2}+\left(\dot{R}\right)^2=\frac{d}{dt}\left(2R\dot{R}\right)+R^2=2R^2+2R\ddot{R}+\dot{R}^2$

Now, (4) becomes $p=\pi+\frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2}+R^2\right]$ ... (5)

Second Part :

Let R = a (2 + Cos nt) ... (6)

Let there be no cavitation in the fluid every where on the surface so that p>0.

Then we have to prove that $\pi > 3\rho a^2 n^2$

We have $\dot{R} = -an \sin nt$

$\ddot{R} = -an^2 \cos nt$

We observe that

$$2R\ddot{R} + 3\dot{R}^2$$

$$= 2a(2+\cos nt)(-an^2 \cos nt) + 3a^2n^2 \sin^2 nt$$

$$= a^2n^2\left[-4\cos nt - 2\cos^2 nt + 3\sin^2 nt\right]$$

$$= a^2n^2\left[-4\cos nt - 2 + 5\sin^2 nt\right]$$

Using this in (4) we have

$$p = \pi + \frac{1}{2}\rho a^2 n^2\left[-4\cos nt - 2 + 5\sin^2 nt\right] \quad ... (7)$$

As cosnt varies from -1 to 1 and so R varies from a to 3a by (6). Thus the sphere shrinks from R=3a to R = a, and so there is a possibility of cavitation. Also p is minimum when nt=0 or 2mπ.

$$p_{min} = \pi + \frac{1}{2}\rho a^2 n^2[-4-2+0] \text{ by (7) } p = \pi - 3\rho a^2 n^2$$

$$p > 0 \Rightarrow p_{min} > 0 \Rightarrow \pi - 3\rho a^2 n^2 > 0 \Rightarrow \pi > 3\rho a^2 n^2 .$$

Exercise 4 :

A mass of liquid surrounds a solid sphere of radius a and its outer surface, which is concentric sphere of radius b, is subject to a given constant pressure π, no other forces being in action on the liquid. The solid sphere is suddenly shrinks into a concentric sphere; it is required to determine the subsequent motion and impulsive action on the sphere.

Solution :

Equation of motion is $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ ... (1)

Equation of continuity is $x^2 u = F(t)$ ... (2)

Hence, $\dfrac{F'(t)}{x^2} + \dfrac{\partial}{\partial t}\left(\dfrac{1}{\rho}u^2\right) = -\dfrac{\partial}{\partial t}\left(\dfrac{p}{\rho}\right)$

Integrating w.r.t. x we get

$$\frac{-F'(t)}{x^2} + \frac{1}{\rho}u^2 = -\frac{p}{\rho} + C \quad \text{... (3)}$$

Since the liquid contained between two spheres r = a, r=b, so we suppose that r and R are internal and external radii at any time t and the corresponding velocities are u and U respectively. Boundary conditions are -

$$x = \gamma \qquad u = v = \dot{r} \qquad p = 0 \quad \text{... (4)}$$

$$x = R \qquad u = \ddot{R} = U \qquad p = \pi \quad \text{... (5)}$$

$$r = a, \qquad u = \dot{r} \text{ so that } F(t) = 0 \quad \text{... (6)}$$

Subjecting (3) to the boundary conditions (4) and (5)

$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = 0 + C = -\frac{p}{\rho} + \frac{\pi}{\rho}$$

$$\frac{-F'(t)}{R} + \frac{1}{2}U^2 = -\frac{\pi}{\rho} + C$$

Also, $r^2u = F(t) = R^2U$ upon subtraction

$$F'(t)\left\{\frac{1}{R} - \frac{1}{r}\right\} + \frac{1}{2}F^2\left\{\frac{1}{r^4} - \frac{1}{R^4}\right\} = \frac{\pi}{\rho} \quad \text{... (7)}$$

Since $r^2u = F(t) = R^2U$

i.e. $r^2dr = F(t)dt = R^2dR$

Multiplying (7) by $2F(t)dt = 2r^2dr = 2R^2dR$, we get

$$2FF'\left\{\frac{1}{R} - \frac{1}{r}\right\}dt + F^2\left\{\frac{dr}{r^2} - \frac{dR}{R^2}\right\} = \frac{\pi}{\rho}2r^2dr$$

$$d\left[\left(\frac{1}{R} - \frac{1}{r}\right)F^2\right] = \frac{\pi}{\rho}2r^2dr$$

Integrating $\left(\dfrac{1}{R} - \dfrac{1}{r}\right)F^2 = \dfrac{2}{3}\dfrac{\pi}{\rho}r^3 + A$

Using boundary condition (6) $0 = \dfrac{2}{3}\dfrac{a^3}{\rho}\pi + A$

Hence $\left(\frac{1}{R}-\frac{1}{r}\right)F^2(t)=\frac{2\pi}{3\rho}\left(r^3-a^3\right)$

$$\left(\frac{R-r}{rR}\right)\left(r^2\dot{u}\right)^2=\frac{2\pi}{3\rho}\left(a^3-r^3\right)$$

or $$r^3u^2\left(\frac{R-r}{R}\right)=\frac{2\pi}{3\rho}\left(a^3-r^3\right) \qquad \text{... (8)}$$

$$R^3 - r^3 = b^3 - a^3$$

Volume of liquid at time t = volume of liquid intially

$$\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3=\frac{4}{3}\pi b^3-\frac{4}{3}\pi a^3$$

Next to determine the equation of impulsive action.

Equation of impulsive action is

$$d\tilde{w}=\rho u dx=\frac{\rho F}{x^2}dx$$

$$\int_0^{\tilde{w}} d\tilde{w}=\int_r^R \frac{\rho F}{x^2}dx$$

$$=-\rho F\left\{\frac{1}{R}-\frac{1}{r}\right\}$$

$$=\rho r^2 u\left\{\frac{1}{r}-\frac{1}{R}\right\}$$

$$=\rho r^2 u\left(\frac{1}{r}-\frac{1}{R}\right)$$

The whole impulsive on the surface of the sphere is

$$4\pi r^2\tilde{w}=4\pi r^2\rho r^2 u\left(\frac{1}{r}-\frac{1}{R}\right)$$

$$=4\pi r^2\rho u\left(\frac{R-r}{R}\right) \qquad \text{... (9)}$$

(8) and (9) are the required equation.

2.7 Check your progress :

Q 1 : Why is the Euler's equation of motion called momentum equation?

Q2 : What would be the shape of Euler's equation of motion if graviational force is taken into account?

Hints : Put $\vec{F} = \vec{g}\rho$ in the equation

$$\rho\left\{\frac{\partial \overline{q}}{\partial t} + (\overline{q}.\nabla)\overline{q}\right\} = \vec{F} - \nabla p$$

Q3 : From what principle Euler's equation of motion is derived?

Q4 : Under what flow condition we can derive Bernnoull's equation?

Q5 : What is the Bernnoull's equation in steady motion?

Q6 : Write down Euler's equation of motion

$$\rho\left\{\frac{\partial \overline{q}}{\partial t} + (\overline{q}.\nabla)\overline{q}\right\} = \vec{F} - \nabla p$$

in scaler form in terms of following co-ordinate systems

(i) rectangular,

(ii) spherical and

(iii) cylinderical co-ordinate.

Q7 : If fluid flow is $\overline{q} = u(y)i$ what would be the shape of governing equation. $\hat{z}^{/}$ is take vertically upwards. The gravity force is acting downwards.

Hints : take the three components of

$$\rho\left\{\frac{\partial \overline{q}}{\partial t} + (\overline{q}.\nabla)\overline{q}\right\} = -g\hat{z}^{/}\rho - \nabla p$$

you will get $\rho\dfrac{du(y)}{dt} = \dfrac{\partial p}{\partial x}$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -g\rho$$

2.8 Let us sum up :

The principle of conservation of momentum gives the governing equation of motion. Here in this particular case we consider the ideal fluid where the viscous effect is not taken into account. Such a governing equation for fluid flow is known as Euler's equation of motion.

If the flow is of potential kind and external forces are conservative Euler's equation takes a particular form which is known as Bernoulli's equation.

When there is sudden change of velocities at the boundaries under action of impulsive force the governing equation takes a special shape. This is known as governing equation for impulsive motion.

If $\overline{q}$ is fluid velocity then the line integral along closed contour $\int_C \overline{q}.d\overline{\rho}$ is known as circulation round the curve C. Kelvin's theorem states that under certain conditions in a moving fluid circulation may be found to be a constant.

BLOCK - 3 : GENERAL THEORY OF IRROTATIONAL MOTION

Structure

3.0 **Objective** :

In this unit we will be concerned with incompressible irrotational flow theory. Green's theorem will be proved and applications of this theorem in potential flow investigations will be made. Expressions for Kinetic energy for the potential flow in various cases will be obtained and some uniqueness theorems related to acyclic irrotational motion will be proved.

3.1 **Introduction** :

The law of conservation of circulation introduced in block 2 gives us an important result.

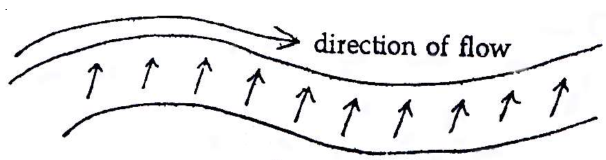
Let us consider a streamline. We find that $\nabla x\, \overline{q}$ is zero at some point on it. We draw an arbitrary infinitely small closed contour to encircle the steamline at that point. In the course of time contour moves with the fluid but always encircles the every point on the steramline. Since $\int_C \overline{q}.d\overline{r} = \text{constant}$ (as obtained in Block -2) it follows that $\nabla x\, \overline{q} = 0$ at every point on the streamline.

If the flow is not steady, the same result holds, except that instead of streamline we must consider the path described in the course of time by some particular fluid particles.

If the flow velocity $\overline{q} = \text{constant}$ so that $\nabla x\, \overline{q} = 0$ on all streamlines. The concept of potential flow is developed from such aspect fluid flow.

3.2 **Potential Flow** :

A flow is called irrotional or of the potential kind if curl $\vec{q} = 0$. In such a flow if we imagine that the liquid at a point solidifies instantaneously, then this solid will have only translational velocity. Thus in irrotional motion there will be no angular velocity. In such a flow the velocity vector can always be expressed as gradient of a scalar function ϕ i.e. $\vec{q} = -\vec{\nabla}\phi$. The function ϕ, known as velocity potential, is single valued if the region of flow is simply connected.



Irrotational flow

The equation of motion for irrotional flow of an incompressible fluid can be integrated resulting Bernoulli's equation and the equation of continuity will give

$$\vec{\nabla}.\vec{q} = 0 = \vec{\nabla}.\left(-\vec{\nabla}\phi\right) = 0 \Rightarrow \vec{\nabla}^2\phi = 0 \qquad \text{... (3.1)}$$

Which implies that for an irrotational flow of an incompressible fluid the velocity potential sat isfies Laplace;s equation. When the region of flow is finite, equation (2.1) can be solved if $\frac{\partial \phi}{\partial n}$ is prescribed at the boundaries. In fluid flow $-\frac{\partial \phi}{\partial n}$ i.e. the normal component of the velocity is generally prescribed at the boundaries. When the fluid is infinite in extent and a solid boundary is present in it, the equation (3.1) can be solved uniquely when

i) $\phi \to \phi_\alpha$ for large values of x.

ii) $\frac{\partial \phi}{\partial n}$ is prescribed on the solid surface. The equation (3.1) can be written in curvilinear coordinates as

$$\frac{\partial}{\partial x_1}\left[\frac{h_2 h_3}{h_1}.\frac{\partial \phi}{\partial x_1}\right]+\frac{\partial}{\partial x_2}\left[\frac{h_3 h_1}{h_2}.\frac{\partial \phi}{\partial x_2}\right]+\frac{\partial}{\partial x_3}\left[\frac{h_1 h_2}{h_3}.\frac{\partial \phi}{\partial x_3}\right]=0$$

3.3 Green's Theorem and its applications :

3.3.1 Green's Theorem :

If ϕ and $\phi^{/}$ are single valued continuously differentiable scalar point functions such that $\vec{\nabla}\phi$ and $\vec{\nabla}\phi^{/}$ both are also continuously differentiable, then

$$\int_V \left(\vec{\nabla}\phi.\vec{\nabla}\phi^{/}\right) dV = -\int_S \phi \frac{\partial \phi^{/}}{\partial n} ds - \int_V \phi \nabla^2 \phi^{/}\, dV \qquad \ldots (3.2)$$

$$= -\int_S \phi^{/} \frac{\partial \phi}{\partial n} ds - \int_V \phi^{/}\, \nabla^2 \phi\, dV$$

where s is closed surface bounding a simply connected region, δn is an element of inward normal at a point on S, and V is the volume enclosed by the surface S.

Proof : Let $\vec{A}$ be a vector field and $\hat{n}$ be a unit vector in the direction of the inward normal, then Gauss theorem gives

$$\int_S \left(\hat{n}\right).\vec{A}\, ds = \int_V \vec{\nabla} A\, dV$$

Putting $\vec{A} = \phi \vec{a}$ in the above equation, we get

$$-\int_S \hat{n}.\left(\phi \vec{a}\right) ds = \int_V \vec{\nabla}.\left(\phi \vec{a}\right) dV$$

$$= \int_V \phi.\left(\vec{\nabla}.\vec{a}\right)dV + \int_V \left(\vec{\nabla}\phi\right).\vec{a}\,dV$$

Putting $\vec{a} = \vec{\nabla}\phi'$ in the above equation, we get

$$-\int_S \hat{n}.\left(\phi\vec{\nabla}\phi'\right)ds = \int_V \phi.\left(\vec{\nabla}.\vec{\nabla}\phi'\right)dv + \int_V \left(\vec{\nabla}\phi\right).\left(\vec{\nabla}\phi'\right)dV$$

$$\Rightarrow -\int_S \phi\frac{\partial\phi'}{\partial n}ds = \int_V \phi.\vec{\nabla}^2\phi' dv + \int_V \left(\vec{\nabla}\phi\right).\left(\vec{\nabla}\phi'\right)dV$$

$$\Rightarrow \int_V \left(\vec{\nabla}\phi\right).\left(\vec{\nabla}\phi'\right)dV = -\int_S \phi\frac{\partial\phi'}{\partial n}ds - \int_V \phi.\nabla^2\phi' dv \quad .\ (3.3)$$

Interchanging ϕ and ϕ' is above equation and noting that

$\left(\vec{\nabla}\phi'\right).\left(\vec{\nabla}\phi\right) = \left(\vec{\nabla}\phi\right).\left(\vec{\nabla}\phi'\right)$, we get

$$\int_V \left(\vec{\nabla}\phi\right).\left(\vec{\nabla}\phi'\right)dv = -\int_S \phi'\frac{\partial\phi}{\partial n}ds - \int_V \phi'.\nabla^2\phi dv \quad \ldots (3.4)$$

Equation (3.3) and (3.4) together proves the Green's theorem.

3.3.2 **Application of Green's Theorem in fluid Dynamics** :

We shall now deduce some results of fluid dynamics from Greens theorem.

(i) Putting ϕ'= constant and taking ϕ to be the velocity potential of a motion of an incompressible fluid, from Green's theorem (3.2) we get,

$$O = -\int_S \frac{\partial\phi}{\partial n}ds - \int_V \nabla^2\phi dv \quad \ldots (3.5)$$

Since we know that in a potential flow of an incompressible fluid velocity potential satisfies Laplace's Equation

i.e. $\nabla^2\phi = 0$ hence (3.5) yields

$$\int_S \frac{\partial\phi}{\partial n}ds = 0 \quad \ldots (3.6)$$

Which means that the total flow of an incompressible fluid into any closed region at any instant is zero.

(ii) If ϕ and ϕ' be the velocity potentials of two liquid motions taking place in the region bounded by the surface S. then $\nabla^2\phi = \nabla^2\phi' = 0$ and hence Green'ss Theorem (3.2) yields

$$\int_S \phi \frac{\partial \phi'}{\partial n} ds = \int_S \phi' \frac{\partial \phi}{\partial n} ds$$

$$\Rightarrow \int_S \rho\phi \left(-\frac{\partial \phi'}{\partial n}\right) ds = \int_S \rho\phi' \left(-\frac{\partial \phi}{\partial n}\right) ds \qquad \text{... (3.7)}$$

Since $-\frac{\partial \phi'}{\partial n}$ is the normal velocity inwards and $\rho\phi$ is the impulsive pressure at any point on the surface which will produce velocity potential ϕ from rest hence (3.7) has the physical meaning that if there be two possible motion inside the surface S by means of two diffeent impulsive pressures on the boundary then work done by the first in acting through the displacement produced by the second must be equal to the work done by the second in acting through the displacement produced by the first.

(iii) The case when $\phi = \phi'$ is a velocity potential of a liquid motion within S then $\nabla^2\phi = 0$ and hence Green's theorem (3.2) gives

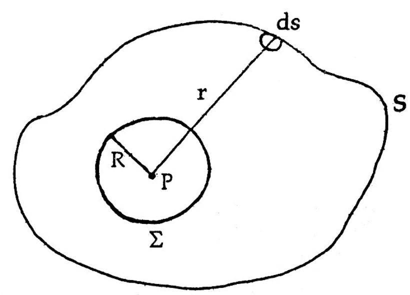
$$\int_V (\nabla\phi)^2 dv = -\int_S \phi \frac{\partial \phi}{\partial n} ds$$

If $\vec{q}$ be the velocity and ρ the density of the liquid then above equation reduces to

$$\frac{1}{2}\rho \int_V (q)^2 dv = \frac{-1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n} ds \qquad \text{... (3.8)}$$

Equation (3.8) is the statement that the kinetic energy set up by impulses, in a system from rest, is the sum of the products of each impulse and half the velocity of its point of application. This result also shows that the kinetic energy of a given mass of a liquid moving irrotionally depends only on the motion of its boundaries. If the boundaries are at rest equation (3.8) gives that velocity should be zero everywhere in the region. Thus irrotational motion is impossible in a closed region with fixed boundaries.

(iv) Let S be a closed surface at every point of whose interior $\nabla^2\phi = 0$. Let P be a point interior to S and let r be the distance of the point P from the element of area ds. Draw a sphere $\sum$ with centre P and radius R. The region $\sum$ should lie entirely within S as shown in the figure



If we take $\phi' = \frac{1}{r}$ the it can easily be verified that $\nabla^2\phi' = 0$.

Further, if ϕ and ϕ' satisfy $\nabla^2\phi = 0 = \nabla^2\phi'$ then from Green's theorem (3.2) we get,

$$\int_V \left(\vec{\nabla}\phi\right).\left(\vec{\nabla}\phi'\right) dV = -\int_S \phi \frac{\partial\phi'}{\partial n}\, ds$$

and $$\int_V \left(\vec{\nabla}\phi\right).\left(\vec{\nabla}\phi'\right) dV = -\int_S \phi' \frac{\partial\phi}{\partial n}\, ds$$

Since $\left(\vec{\nabla}\phi\right).\left(\vec{\nabla}\phi'\right) = \left(\vec{\nabla}\phi'\right).\left(\vec{\nabla}\phi\right)$ hence the above two equations give

$$\int_S \left(\phi \frac{\partial \phi'}{\partial n} - \phi' \frac{\partial \phi}{\partial n} \right) ds = 0 \qquad \text{... (3.9)}$$

Here S stands for the total surface.

For the case when ϕ is a velocity potential and $\phi' = \frac{1}{r}$ the equation (3.9) yields,

$$-\int_S \left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} \right] ds + \int_\Sigma \left[\phi \frac{\partial}{\partial R}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial \phi}{\partial R} \right] d\Sigma = 0 \quad .(3.10)$$

Since the direction of the normal on the sphere is that of r put in the opposite sense hence dn has been replaced by (-dR).

The terms $\int_\Sigma \frac{\partial \phi}{\partial R} d\Sigma = \int_\Sigma \vec{\nabla}\phi . \hat{R}\, d\Sigma = \left[\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial \phi}{\partial R} \right] = \int_V \vec{\nabla}\left(\vec{\nabla}\phi\right) dV = \int_V \nabla^2 \phi\, dV = 0$

If we take R so small that ϕ is constant in the space $\sum$ and its value is ϕ_p then the term.

$$\int_\Sigma \phi \frac{\partial}{\partial R}\left(\frac{1}{R}\right) d\Sigma = \int_\Sigma \phi \left(-\frac{1}{R^2} \right) d\Sigma$$

$$= -\phi_P . \frac{1}{R^2} . \int_\Sigma d\Sigma$$

$$= -\phi_P . \frac{1}{R^2} . \Sigma$$

$$= -\phi_P . \frac{1}{R^2} . 4\pi R^2$$

$$= -4\pi \phi_P$$

Taking the limit of the equation (3.10) as R$\to$0, we get,

$$\phi_P = \frac{1}{4\pi} \int_S \left[-\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{\partial \phi}{\partial n} \right] ds \qquad . . . (3.11)$$

In the case when S is a sphere of radius r, the equation (3.11) gives

$$\phi_P = \frac{1}{4\pi}\int_S \left[-\phi \frac{\partial}{\partial r}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{\partial \phi}{\partial r} \right] \text{ds}$$

$$\phi_P = \frac{1}{4\pi r^2}\int_S \phi ds + \frac{1}{4\pi r}\int_S \frac{\partial \phi}{\partial r} \text{ ds} \qquad ...(3.12)$$

Applying Gauss theorem to the second integral, we get,

$$\int_S \frac{\partial \phi}{\partial r} ds = \int_S \vec{\nabla}\phi.\hat{r} ds = \int_V \vec{\nabla}.(\vec{\nabla}\phi) \text{ dv} = \int_V \nabla^2 \phi \text{dV} = 0$$

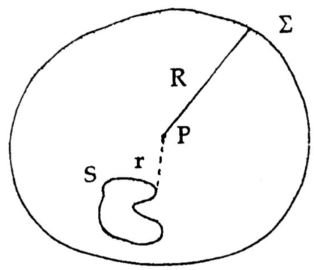
Hence from equation (3.12), the value of Φ_p is given by

$$\phi_P = \frac{1}{4\pi r^2}\int_S \phi ds \qquad ...(3.13)$$

The equation (3.13) is the statement that the mean value of Φ over any spherical surface, throughout whose interior $\nabla^2\Phi = 0$ is equal to the value of Φ at the centre of the sphere.

From equation (3.13) it can also be deduced that Φ cannot be a maximum or minimum in the interior or any region throughout which $\nabla^2\Phi = 0$. For if Φ_p be a maximum value of Φ at a point P then it would be greater than the value of Φ at all points of a sufficiently small sphere centered at P, which contradicts the above theorem.

(v) In the case when the region of flow is outside a solid body of surface S we take a point P in the liquid outside the solid surface S and consider a sphere of large radius R with P as the centre in which the region S is enclosed as shown in the figure.



Taking the space between S and Σ and using equation (3.11), we get the value of Φ_p as

$$4\pi\phi_p = \int_S \left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n}\right] ds = \int_\Sigma \left[\phi \frac{\partial}{\partial R}\left(\frac{1}{R}\right) - \frac{1}{R}\frac{\partial \phi}{\partial R}\right] d\Sigma \quad ..(3.14)$$

To get the value of the second term on the right hand side of above equation, we apply Gauss theorem on $\vec{\nabla}\phi$ by taking the region of integration as the space between S and Σ and get

$$\int_S \frac{\partial \phi}{\partial n} ds + \int_\Sigma \frac{\partial \phi}{\partial n} d\Sigma = -\int_V \nabla^2 \phi dV$$

$$\Rightarrow \int_S \frac{\partial \phi}{\partial n} ds + \int_\Sigma \frac{\partial \phi}{\partial n} d\Sigma = 0 \qquad (\because \nabla^2\phi = 0)$$

Since the flow across S is zero therefore

$\int_S \frac{\partial \phi}{\partial n} ds = 0$ and hence the above equation gives

$$\int_\Sigma \frac{\partial \phi}{\partial R} d\Sigma = 0 \qquad ... (3.15)$$

Writing $dS=R^2dw$, the above equation becomes

$$\int_\Sigma \frac{\partial \phi}{\partial R} d\omega = 0$$

$$\Rightarrow \frac{\partial}{\partial R}\int_\Sigma \phi \, d\omega = 0$$

$$\Rightarrow \int_\Sigma \phi \, d\omega = 4\pi c \qquad ... (3.16)$$

The constant c is independent of R, the constant c is independent of the point P. we displace the point by a small distance δx keeping R constant. Then the equation (3.1 6) gives

$$\int_{\Sigma} \frac{\partial \phi}{\partial x}\, d\omega = 4\pi \frac{\partial c}{\partial x} \qquad \text{... (3.17)}$$

Since the liquid is at rest at infinity, $\frac{\partial \phi}{\partial x} = 0$ on Σ when $R \to \alpha$. Hence for large value of R, (3.17) shows that $\frac{\partial c}{\partial x} = 0$.

Thus we see that C is an absolute constant.

Using the equations (3.15) and (3.16), the value of ϕ_p is given by (3.14) as

$$4\pi\left(\phi_p - c\right)\left[\phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n}\right] ds \qquad \text{... (3.18)}$$

If $r \to \propto$ then $\frac{1}{r}$ as well as $\frac{\partial}{\partial n}\left(\frac{1}{r}\right)$ tend to zero. Hence from equation (3.18) it can be interpreted that $\phi_p \to C$ as $r \to \propto$

We have given above some deductions from Green's theorem, but we remark that many of these are capable of very simple independent proof.

3.4 **Kinetic energy of a Liquid Extending to infinity** :

It has been given earlier that the kinetic energy of liquid contained in a closed surface S is

$$T = \frac{1}{2}\rho \int_V q^2 dV = -\frac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n}\, dS$$

The case when the liquid is at rest at infinity and is bounded internally by a solid surface S, we consider the liquid in the space between this soild and sphere of a large radius and then the Kinetic energy is given by

$$T = \frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,dS - \frac{1}{2}\rho\int_\Sigma \phi\frac{\partial\phi}{\partial n}\,d\Sigma \qquad \text{... (3.19)}$$

There is no flow in the region across S, hence from the equation of continuity, we get,

$$\int_\Sigma \phi\frac{\partial\phi}{\partial n}\,d\Sigma = 0 \qquad \text{... (3.20)}$$

Multiplying (3.20) by $\frac{1}{2}C$, the the limiting value of ϕ at infinity and adding it to the equation (3.19), we get,

$$T = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,dS - \frac{1}{2}\rho\int_\Sigma (\phi - c)\frac{\partial\phi}{\partial n}\,d\Sigma$$

The second Integral tends to zero as the radius of the sphere S tends to infinity, hence the kinetic energy, in this case is

$$T = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,dS \qquad \text{... (3.21)}$$

This shows that irrotational motion is impossible in a liquid which is rest at infinity and is bounded internally by fixed rigid walls. It can also be interpreted from equation (3.21) that the case in which liquid is infinite and solids are moving in it such that the motion is irrotational, the motion will stop immediately if the solids are brought to rest.

3.5 Kelvin's minimum energy theorem :

The irrotational motion of a liquid occupying a simply connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary.

Proof : Let T_1 be the kinetic energy and $\vec{q}_{1}$, be the fluid velocity of the actual irrotational motion with a velocity potential Φ. Then

$$\vec{q}_1 = -\vec{\nabla}\Phi \qquad \text{... (3.22)}$$

Let T_2 be the kinetic energy and $\vec{q}_2$ be the fluid velocity of any other possible state of motion consistent with the same normal velocity of the boundary S.

The equation of continuity, for the above two motion, gives

$$\vec{\nabla}\cdot\vec{q}_1 = 0 \text{ and } \vec{\nabla}\cdot\vec{q}_2 = 0 \qquad \text{... (3.23)}$$

Let $\hat{n}$ denote the unit normal vector at a point P on S. Then, since the boundary has the same normal velocity in both motions, we have

$$\hat{n}.\vec{q}_1 = \hat{n}.\vec{q}_2$$

Now, $T_1 = \frac{1}{2}\rho\int_V q_1^2 dV = \frac{1}{2}\rho\int_V \vec{q}_1^{\,2}\, dV$

and $T_2 = \frac{1}{2}\rho\int_V q_2^2 dV = \frac{1}{2}\rho\int_V \vec{q}_2^{\,2}\, dV$

therefore,

$$T_2 - T_1 = \frac{1}{2}\rho\int_V \left(\vec{q}_2^{\,2} - \vec{q}_1^{\,2}\right) dV$$

$$= \frac{1}{2}\rho\int_V \left[2\vec{q}_1.\left(\vec{q}_2 - \vec{q}_1\right) + \left(\vec{q}_2 - \vec{q}_1\right)^2\right] dV$$

(note carefully this step)

$$= \rho\int_V \overline{q_1}.\left(\vec{q}_2 - \vec{q}_1\right) dV + \frac{1}{2}\rho\int_V \left(\vec{q}_2 - \vec{q}_1\right)^2 dV$$

$$T_2 - T_1 = -\rho\int_V \left(\vec{\nabla}\phi\right).\left(\vec{q}_2 - \vec{q}_1\right) dV + \frac{1}{2}\rho\int_V \left(\vec{q}_2 - \vec{q}_1\right)^2 dV \qquad \text{.. (3.25)}$$

Since we know that

$$\vec{\nabla}\left[\phi.\left(\vec{q}_2 - \vec{q}_1\right)\right] = \phi\vec{\nabla}.\left(\vec{q}_2 - \vec{q}_1\right) + \left(\vec{\nabla}\phi\right).\left(\vec{q}_2 - \vec{q}_1\right)$$

$$= \left(\vec{\nabla}\phi\right).\left(\vec{q}_2 - \vec{q}_1\right) \qquad \text{(by using 3.23)}$$

Therefore $\int_V \left(\vec{\nabla}\phi\right).\left(\vec{q}_2 - \vec{q}_1\right) dV = \int_V \vec{\nabla}.\left[\phi\left(\vec{q}_2 - \vec{q}_1\right) dV\right]$

$$= \int_V \phi\left(\vec{q}_2 - \vec{q}_1\right).\hat{n}ds$$

$$= 0 \qquad \text{... (3.26)}$$

(by using (3.24)

Using (3.26) in (3.25), we get

$$T_2 - T_1 = -\frac{1}{2}\rho\int_V \left(\vec{q}_2 - \vec{q}_1\right)^2 dV \qquad \text{... (3.27)}$$

Since the R.H.S of (3.27) is non-negative, hence

$$T_2 - T_1 \geq 0$$

$$\Rightarrow T_2 \geq T_1 \Rightarrow T_1 \geq T_2$$

This proves the theorem.

3.6 Uniqueness Theorems :

We shall now prove some therorems concerning acyclic motion (the motion in which the velocity potential is single-valued is called acyclic) of a liquid by making use of the following equivalence of the expressions for the kinetic energy,

$$T = \frac{1}{2}\rho\int_V q^2 dV = -\frac{1}{2}\rho\int_V \phi\frac{\partial\phi}{\partial n}\, dS \qquad \text{... (3.28)}$$

Where V is the volume of the liquid ecnlosed by the boundary surface S.

Theorem 1. There can not be two different forms of acyclic irrotational motion of a confined mass of liquid in which the boundaries have prescribed velocities.

Proof : If possible, let ϕ_1 and ϕ_2 be the velocity potentials of two different acyclic motions subject to the condition

$$\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n} \text{ at each point of the boundary S.} \qquad .... (3.29)$$

Since ϕ_1 and ϕ_2 are velocity potentials

$$\therefore \nabla^2 \phi_1 = \nabla^2 \phi_2 \text{ at each point of V.} \qquad .... (3.30)$$

Let $\phi = \phi_1 - \phi_2$

Then $\nabla^2 \phi = \nabla^2 (\phi_1 - \phi_2) = \nabla^2 \phi_1 - \nabla^2 \phi_2$

$$\Rightarrow \nabla^2 \phi = 0 \text{ at each point of V.} \qquad .... (3.31)$$

[by using (3.30)]

Hence ϕ is a solution of Lapace's equation and so it represents irrotational motion of liquid in which

$$\frac{\partial \phi}{\partial n} = \frac{\partial}{\partial n}(\phi_1 - \phi_2) = \frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n}$$

i.e. $$\frac{\partial \phi}{\partial n} = 0 \text{ at each point of S} \qquad ... (3.32)$$

[by using (3.29)]

The Kinetic energy for the liquid with $\phi = \phi_1 - \phi_2$ will be given by

$$T = -\frac{1}{2}\rho \int_V \phi \frac{\partial \phi}{\partial n} \, dS$$

which will be zero because of the boundary conditon (3.32)

Hence T = 0

$$\Rightarrow -\frac{1}{2}\int_S q^2 dV = 0 \qquad \text{[by using (3.28)]}$$

$$\Rightarrow \rho \int_S q^2 dV = 0$$

$\Rightarrow q^2 = 0$ (at each point of V)

$\Rightarrow q = 0 \Rightarrow \vec{q} = 0$

$\Rightarrow \vec{\nabla}\phi = 0$

$\Rightarrow \phi = \text{constant}$

$\Rightarrow \phi_1 - \phi_2 = \text{constant}$

Since the motion is acyclic therefore ϕ_1 and ϕ_2 are single valued. Further the constant is of no significance hence above equation reveals that the two motions are essential the same.

Theorem 2 : If given impulsive pressures are applied to the boundaries of a confiend mass of loquid at rest, the resulting motion, if a cyclic and irrotational will be uniquey determined.

Proof : If possible, ϕ_1 and ϕ_2 be velocity potentials of two diferent irotational motion. Let the impulsive pressure which would start the first motion is $\rho\phi_1$, that which would start the second is $\rho\phi_2$. It is given that

$$\rho\phi_1 = \rho\phi_2 \text{ at each point of boundary S.} \qquad \text{... (3.33)}$$

Since ϕ_1 and ϕ_2 are velocity potentials of irrotational motions, therefore,

$$\nabla^2\phi_1 = 0 = \nabla^2\phi_2 \text{ at each point of V.} \qquad \text{... (3.34)}$$

Let $\phi = \phi_1 - \phi_2$

Then $\rho\phi = \rho(\phi_1 - \phi_2) = \rho\phi_1 - \rho\phi_2$

$$\Rightarrow \rho\phi = 0 \text{ at each point of S} \qquad \text{... (3.35)}$$

[by making use of (3.33)]

Further $\nabla^2\phi = \nabla^2(\phi_1 - \phi_2) = \nabla^2\phi_1 - \nabla^2\phi_2$

$$\Rightarrow \nabla^2\phi = 0 \text{ at each point of V.} \qquad \text{... (3.36)}$$

[by making use of (3.34)]

Hence ϕ is a solution of Laplace's equation and so it represents irrotational motion of a liquid.

The Kinetic energy for the liquid with $\phi = \phi_1 - \phi_2$ will be given by

$$T = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,dS = -\frac{1}{2}\int_S (\rho\phi)\frac{\partial\phi}{\partial n}\,dS$$

which will be zero because of the boundary condition (3.35)

Hence T = 0

Now proceeding in the same way as in the proof of Theorem I, given above, we can conclude that the two motions are essentially the same.

Theorem 3 : Acyclic irrotational motion is impossible in a liquid bounded entirely by fixed rigid walls.

Proof : Since $\frac{\partial\phi}{\partial n} = 0$ at every point of the boundary therefore, the Kinetic energy

$$T = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,dS$$

$$\Rightarrow T = 0 \qquad (\because \frac{\partial\phi}{\partial n} = 0 \text{ everywhere on the wall})$$

$$\Rightarrow -\frac{1}{2}\rho\int_V q^2 dV = 0$$

$$\Rightarrow \int_V q^2 dV = 0$$

Since q^2 can not be negative, therefore

q = 0 everywhere in V

and hence we conclude that the liquid is at rest.

3.7 Check your Progress :

1. Define irrotational motion and prove that under certain conditions the motion of a frictionless liquid, if once irrotational, is always so. Prove that this theorem remains true when each particle is acted on by a resistance varying as the velocity.
2. If the velocity Potential ϕ is constant over the boundary of any simply connected region occupied by liquid in irrotational motion, prove that ϕ has the same constant value throughout the interior.
3. If the normal velocity is zero at every point of the boundary occupying a simply connected region, and moving irrotationally then prove that the velocity potential is constant throughout the interior of that region.
4. A space is bounded by an ideal fixed surface S drawn in a homogeneous incompressible fluid satisfying the conditions for the continued existence of a velocity potential ϕ under conservative forces. Prove that the rate per unit time at which energy flows across S into the space bounded by S is

$$-\rho\int_{S}\phi\frac{\partial\phi}{\partial t}.\frac{\partial\phi}{\partial n}\,dS.$$

Here ρ is the density and δn and element of the normal to ds drawn into the space considered.
5. If q is the resultant velocity at any point of a fluid which is moving irrotationally in two dimension, prove that

$$\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=q\nabla^2 q$$

3.8 Points to Remember :

- The necessary and sufficient condition for a flow field to be irrotational is $\vec{\nabla} \times \vec{q} = 0$
- In potential flow of an incompressible fluid, velocity potential satisfies Laplace's equation.
- In Potential flow of an incompressible fluid the expression for kinetic energy T is given by

$$T = -\frac{1}{2}\rho \int_{s} \phi \frac{\partial \phi}{\partial n}\, dS$$

- The motion in which the velocity potential is single-valued is called acyclic motion whereas the motion in which the velocity potential is not single valued is called cyclic.
- A region in which every circuit is reducible is known as simply connected.

BLOCK - 4 : TWO DIMENSIONAL MOTION

UNIT – I : GENERAL THEORY OF IRROTATIONAL MOTION

Structure

BLOCK - 4

TWO DIMENSIONAL MOTION

4.0 Objective :

In this block you will be introduced with two dimensional motion of incompressible fluids. The concept of images will be introduced here. Stream function and complex velocity potential will be introduced and applications of these functions will be discussed in the context of some special two dimensional motions. Some important theorems, viz. Milne Thomson circle theorem, Blasius theorem will be proved. The Blasius theorem to be discussed in this block deals with the estimation of magnitude of the resultant force experienced by a fixed cylinder in presence of circulation. Some of the two-dimensional flow examples will also be introduced to you in this block.

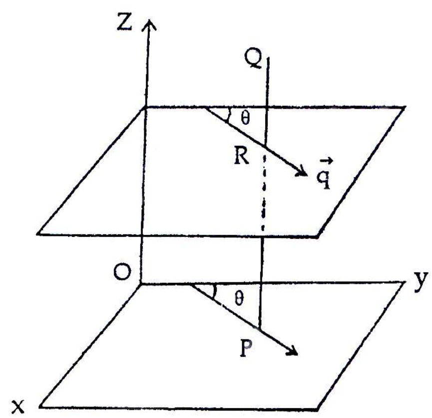
4.1 Introduction :

If a fluid moves in such a way that at any given instant the flow pattern in a certain plane is the same as that in all other parallel planes within the fluid then the flow is said be two dimensional. If we take any one of parallel planes to be the plane z=0, then at any point in the fluid have Cartesian coordinates (x,y,z), all physical quantities (velocity, pressure, Temperature, density etc.) associated with the fluid are independent of z. Thus u,v are functions of x,y and t for such a motion.

The term " Two-dimensional motion" should not be misinterpreted with that motion which is taking place in a two-dimensional space because fluid motion always takes place in three dimension.

4.2 Two Dimensional Motion

A thorough discussion is presented here. let P (x,y,o) be any point in the z plane. Draw PQ perpendicular to the z plane. Then points on the line PQ are said to correspond to the point P. Take any plane (say, at a distance z from z plane) in the fluid parallel to z plane and meeting PQ in R. Then, if the velocity at this point P is $\vec{q}$ in the z plane in a directon θ with OY, the velocity at R (x,y,z) is equal in magnitude and parallel in direction to the velocity at the point P. The velocity at corresponding points is then a function of x,y and t, but not of z.



In order to maintain reality we use to suppose the fluid in two dimensional motion to be confined between two planes parallel to the plane of motion and at unit distance apart. Thus while discussing the flow of a fluid past a cylinder in a two dimensional motion in planes perpendicular to the axis of the cylinder, we restrict our attention to a unit length of cylinder confined between the said planes instead of taking care over the cylinder of infinite length.

Thus in the case of a circular cylinder moving in two dimensions the diagram will show a circle, the cross section of the cylinder. Accordingly, when we speak of the flow across a curve in the z plane, we really mean the flow across unit length of the cylinder whose cross section in the z plane is that curve.

4.3 Stream Function

In two dimensional motion, the velocity is a function of x, y and t only and therefore, the differential equation of stream lines becomes

$$\frac{dx}{u}=\frac{dy}{v} \text{ or, vdx - udy = 0} \qquad (4.1)$$

For an incompressible fluid, the equation of continuity in two dimensional motion is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (4.2)$$

The condition for the equation (4.1) to be exact is

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial x}(-u)$$

which is th esame as (4.2). Hence, vdx - udy is an exact differential, say dψ i.e.,

$$\text{vdx - udy} = d\psi \qquad (4.3)$$

But we know that $d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy$ (4.4)

Comparing (4.3) and (4.4) we get

$$u=-\frac{\partial\psi}{\partial y} \text{ and } v=\frac{\partial\psi}{\partial x} \qquad (4.5)$$

The solution for equation (4.1) is, therefore, ψ =constant. This function ψ is called the stream function or current function. It should be noted that the stream function always exists for all types of two dimensional motions-rotational or irrotational, provided the fluid is incompressible and the flow is continuous.

4.3.1 Properties of Stream Function

(i) Stream function is constant along a stream line.

Stream line is given by $\frac{dx}{u} = \frac{du}{v}$

or vdx - udy = 0

But u = - $\frac{\partial \psi}{\partial y}$, v = $\frac{\partial \psi}{\partial x}$, hence we have

$$\frac{\partial \psi}{\partial x} \text{ dx} + \frac{\partial \psi}{\partial y}, \text{dy} = 0 \text{ or } d\psi = 0$$

Hence ψ = constant (4.6)

(ii) The difference of the values of stream function ψ at two point represents the flux of the fluid across andy curve joining the two points.

(The term flux means the rate of flow)

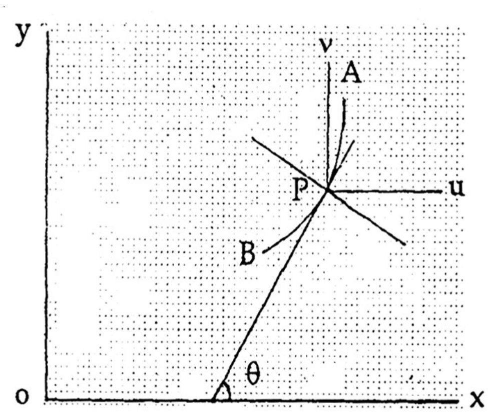
Let ds be an element of a curve AB and θ, the inclination of the tanjent (at a point P of arc ds) to x-axis. Flux of the fluid across AB from left to right is

$\int_A^B$ (flux across and element ds of AB per unit time perunit density)

$$= \int_A^B (u \sin\theta - v\cos\theta) ds$$

$$= \int_A^B (u\frac{dy}{ds} - v\frac{dx}{ds}) \text{ ds}$$

$$= \int_A^B \left(-\frac{\partial \psi}{\partial y} dy - \frac{\partial \psi}{dx} \text{ dx} \right)$$



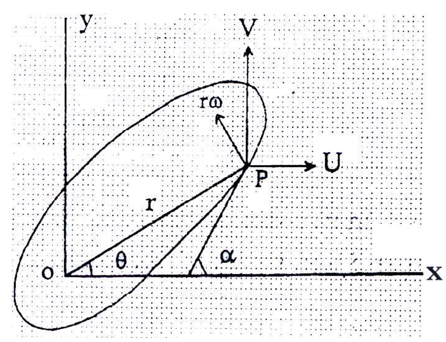
$$-\int_A^B d\psi = \psi_A - \psi_B$$

where ψ_A and ψ_B are respectively the values of ψ at the points A and B respectively.

(iii) **Conditions to be satisfied by ψ at a boundary :**

(a) **At a fixed boundary :** As there is no flow across a fixed boundary hence the fixed boundary is a stream line and since ψ is constant along a stream line hence ψ is constant at a fixed boundary.

(b) **At a moving boundary :** Let motion of the boundary be defined by U,V and ω where U and V are translational velocity components in the directions of x and y respectively and ω be the angular velocity.



The component of the velocity at any point P whose cartesian coordinates are (x,y) an polar coordinates are (r, θ) parallel to the axes of coordinates, are -

$$U - r\omega \sin\theta \quad \text{and} \quad V + r\omega \cos\theta$$

$$\text{or,} \quad U - \omega y \quad \text{and} \quad V + \omega x$$

Boundary condition at the point P is that the normal velocity component of the boundary is equal to the normal velocity component of the fluid particle situated at that point.

That is,

$$(U - \omega y) \sin\alpha - (V + \omega x) \cos\alpha = u \sin\alpha - v\cos\alpha$$

where u, v are the components of fluid velocity at the point P.

$$\text{or} \quad (U - \omega y)\frac{dy}{ds} - (V + \omega x)\frac{dx}{ds} = -\frac{\partial\psi}{\partial y}\frac{dy}{ds} - \frac{\partial\psi}{\partial x}\frac{dx}{ds}$$

or $$U\frac{dy}{ds} - V\frac{dx}{ds} - \omega(y\frac{dy}{ds} + x\frac{dx}{ds}) = -\frac{\partial \psi}{\partial s}$$

In the case when the motion of the boundary is uniform, U,V and ω are constants, hence above equation gives -

$$\psi = Vx - Uy + \frac{1}{2}\omega(x^2 + y^2) + \text{Constant} \qquad (4.7)$$

This is the condition to be satisfied by ψ in case of a boundary which is moving uniformly.

4.4 Irrotational Motion in Two Dimensions

When the motion is irrotational, curl $\vec{q} = 0$ which gives

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \qquad (4.8)$$

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \qquad (4.9)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \qquad (4.10)$$

In two-dimensional motion, (4.8) and (4.9) are automatically zero. Substituting for u and v is terms fo stream function, (4.10) gives -

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \qquad (4.11)$$

which shows that the stream function satisfied Laplace's equation.

Also, in this case, since velocity potential exists we have

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} \qquad (4.12)$$

$$v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} \qquad (4.13)$$

Hence the equation of continuity,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad \text{gives}$$

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=0, \qquad (4.14)$$

Which shows that ϕ also satisfies Laplace's equation.

From (4.12) and (4.13) it can be easily seen that -

$$\frac{\partial \phi}{\partial x}.\frac{\partial \psi}{\partial x}+\frac{\partial \phi}{\partial y}.\frac{\partial \psi}{\partial y}=-uv+uv=0$$

Hence the family of courves ϕ(x,y)=constant and ψ(x,y)=constant cut orthogonally at all their points of intersection.

4.5 Complex potential and Velocity

Let w = $\phi + i\psi$ be taken as a function of x+iy, i.e. of z. Thus, suppose that

$$w=f(z)$$

or $$\phi+i\psi=f(x+iy) \qquad (4.15)$$

Differentiating (4.15) with respect to x and y respectively, we get

$$\frac{\partial \phi}{\partial x}+i\frac{\partial \psi}{\partial x}=f'(x+iy)$$

and $$\frac{\partial \phi}{\partial y}+i\frac{\partial \psi}{\partial y}=i\,f'(x+iy)$$

The above two relation give

$$\frac{\partial \phi}{\partial y}+i\frac{\partial \psi}{\partial y}=i\left(\frac{\partial \phi}{\partial x}+i\frac{\partial \psi}{\partial x}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \tag{4.16}$$

which are Cauchy-Riemann equations. Thus w is an analytic function of z and is known as the complex potential.

Conversly, if w is an analytic function of z, then its real part is the velocity potential and imaginary part is the stream function of an irrotational two dimensional motion.

Again, differentiating the relation,

$$w = \phi + i\psi = f(z)$$

with respect to x, we get

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = f^{/}(z)\frac{\partial z}{\partial x}$$

or, $-u + iv = f^{/}(z)$

or $-u + iv = \frac{dw}{dz}$

Hence $\left|\frac{dw}{dz}\right| = \left|-u + iv\right|$

or $$\left|\frac{dw}{dz}\right| = \sqrt{u^2 + v^2} = \sqrt{\left(\frac{dw}{dz}\right)\overline{\left(\frac{dw}{dz}\right)}} \tag{4.17}$$

Thus $\left(\frac{dw}{dz}\right)$ represents the velocity of any point in an irrotational two dimensional motion.

The points where velocity is zero are called stagnation points. thus, for stagnation points,

$$\frac{dw}{dz} = 0 \tag{4.18}$$

4.5.1 Cauchy-Riemann Equation is polar form

Since we shall very often need the C.R. equations in polar form it is desirable to derive it here.

Let, $\quad w = \phi + i\psi = f(z) = f\left(re^{i\theta}\right)$

Differentiating above equation w.r.t. θ, we get

$$\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r} = f^{/}\left(re^{i\theta}\right)e^{i\theta}$$

Differentiating w.r.t. θ, we get

$$\frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = f^{/}\left(re^{i\theta}\right) r\, i\, e^{i\theta}$$

Since w is an analytic function, it must possess a unique derivative and so,

$$ir\left(\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r}\right) = \frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta}$$

or $$ir\frac{\partial \phi}{\partial r} - r\frac{\partial \psi}{\partial r} = \frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta}$$

Equating real and imaginary parts, we get

$$\frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial r} \text{ and } \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial \theta} \qquad (4.19)$$

which are the polar form of C.R. equations.

4.5.2 Sources, Sinks and Doublets

If the two dimensional motion of a liquid consists of outward radial flow from a point, symmentrical in all directions in the reference plane then the point is called a simple source. So a source in two dimensions will be a line source parallel to z-axis. If the flow across any small closed curve surrounding the source be 2πm then m is called the strength of the source.

A source is thus a point at which fluid is continuously created and distributed. Since the velocity near a source is very large, Bernoulli's theorem demands a large negative pressure. This fact alone shows that a source can have no actual existence and is a purely abstract concept. This concept is useful in discussing the fluid motion in many situations.

A sink is a negative source. Thus a sink is a point of inward flow at which fluid is absorbed or annihilated continuously.

4.5.3 Complex potential due to a source at origin :

If ϕ is the velocity potential due to a source at origin, then

$$2\pi m = 2\pi r\left(-\frac{\partial \phi}{\partial r}\right)$$

$$\Rightarrow \phi = -\text{m} \log \text{r} \qquad (4.20)$$

Since $\frac{\partial \phi}{\partial \text{r}} = \frac{1}{\text{r}}\frac{\partial \phi}{\partial \text{r}}$ hence $\psi = -m\theta$ (4.21)

These values of ϕ and ψ give complex potential

$$w = \phi + i\psi$$

or $$w = -m \log r - im\theta = -m \log \left(re^{i\theta}\right)$$

$$\therefore \text{w} = -\text{m} \log \text{z} \qquad (4.22)$$

If the source is at the point z_0, then by a change of origin

$$\text{w} = -\text{m} \log \left(\text{z} - \text{z}_0\right)$$

Similarly the complex potential for a sink of strength -m is given by

$$w = m \log z \tag{4.23}$$

The complex potential for a system of sources of strength $m_1, m_2, \ldots\ldots, m_r$ at $a_1, a_2, \ldots\ldots, a_r$ and sinks of strength $M_1, M_2, \ldots\ldots, M_s$ at $A_1, A_2, \ldots\ldots, A_s$ is given by

$$w(z) = -\log \frac{\prod_{i}^{r} (z - a_i)^{m_i}}{\prod_{j}^{s} (z - A_j)^{M_j}} \tag{4.24}$$

A doublet is defined as a combination of a source +m and a sink -m at a small distance δs apart such that the product mδs is a finite. If mδs = μ = finit where $m \to \infty$, $\delta s \to 0$ then μ is called strength of the doublet and δs is called the axis of the doublet and its direction is taken from sink to source.

Let a source of strength m be situated at A ($ae^{i\alpha}$) and a sink of strength -m be situated at B($-ae^{i\alpha}$). suppose that A and B are very close to each other so that a is small. Then the complex potential w for this sysem is given by

$$w = -m \log \left(z - ae^{i\alpha}\right) + m \log \left(z + ae^{i\alpha}\right)$$

$$= m \left[\log \left(1 + \frac{ae^{i\alpha}}{z}\right) - \log \left(1 - \frac{ae^{i\alpha}}{z}\right)\right]$$

$$= \frac{2mae^{i\alpha}}{z} + \frac{2ma^3 e^{3i\alpha}}{z^3} + \ldots$$

Let 2ma = μ . Then

$$w = \frac{\mu e^{i\alpha}}{z} + \mu \frac{a^2 e^{3i\alpha}}{z^3} + \ldots.$$

Now let $a \to 0$, μ remaining constant so that $m \to \infty$. Then when A and B coincide, we get

$$w = \frac{\mu e^{i\alpha}}{Z} \tag{4.25}$$

4.5.4 Method of Images

Consider a mass of liquid infinite in extent. suppose there exists two portions P_1 and P_2 of the liquid each having its own proper motion. Suppose further that the two portions have a continuous surface of separation S such that in the motion of the two proportion neither a particle of P_1 nor a particle P_2 traverses S. We say, in such cases, that the motion of one of the portions is the image of the other with respect to S. Under these circumstances, it is possible to suppress completely one of the two portions, say P_2, without affecting the motion of P_1 in any way, provided that the surface S is replaced by a material partition.

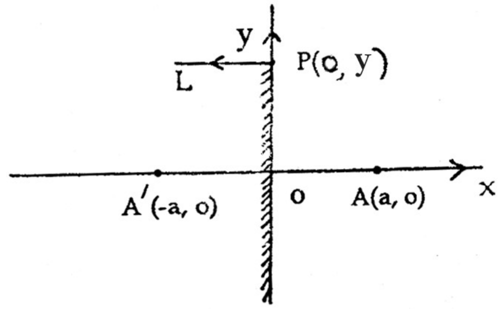
The method of images is used to determine the complex potential due to sources, sinks and doublets in presence of rigid boundaries. Suppose we wish to determine the flow field in one side of a rigid boundary due to sources, sinks and doublets lying in the same side of the boundary. To this end we assume the existence of some hypothetical image sources, sinks and doublets on the opposite side of the boundary in such a manner that the boundary behaves as a stream line or surface.

Thus the given system of sources, sinks and doublets together with the hypothetical one will be equivalent to the given sources, sinks and doublets and the rigid boundaries.

Let us find the Images in some special cases.

(i) **Image of a sources with respect to a line :**

Consider a source m at A(a,o) on x-axis. Suppose we require the image of this source with respect to y-axis. We take an equal source at $A^{/}$(-a,0). Let P be any point on y-axis such that AP = $A^{/}$P = r. Then the velocity at P due to source $A^{/}$ is m/r along $A^{/}$P. Let PL be perpendicular to y-axis.



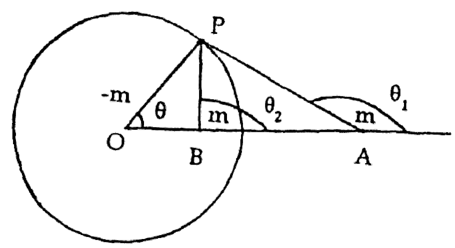
Then resultant velocity at P due to sources at A and A′ along PL

$$=(m/r)\cos\theta-(m/r)\cos\theta=0$$

showing that there will be no flow across y-axis. Hence by definition, the image of a simple source with respect to a line in two dimensions is an equal source equidistant from the line opposite to the source.

(ii) **Image of a source with respect to a circle :**

Let m be the strength of the source at the point A and let B be the inverse point of A with respect to the circle with centre O. Consider a source of strength m at B and an equal sink at O. Then for this arrangement of sources and sink the stream function ψ at any point P on the circle, is given by



$$\psi = -m\theta_1 - m\theta_2 + m\theta$$

or $$\psi = -m(\theta_1 + \theta_2 - \theta) \qquad (4.26)$$

Now, since B is the inverse point of A, we have

$$OA.OB = OP^2$$

or $$\frac{OA}{OP}=\frac{OP}{OB}$$

Hence Δ^s OPA and OBP as similar so that

$$\frac{OA}{OP} = \frac{OP}{OB} = \frac{AP}{PB}$$

and $\angle OPA = \angle OBP$

$\angle OPB = \angle OAP$

and $\angle OPA = \angle OBP$

$\angle OPB = \angle OAP$

i.e. $\theta_2 - \theta = \pi - \theta_1$

$\theta_1 + \theta_2 - \theta = \pi$

Therefore, equation (4.26) becomes

$\psi = -m\pi =$ constant

This shows that ψ is constant upon the circle. The circle is, therefore, a stream line and this verifies that for this arrangement of sources and sink, there is now flow across the boundary.

Hence the image of a source with regard to a cricle is an equal source at the inverse point together an equal sink at the centre of the circle.

4.5.5 Miline-Thomson circle theorem

Let there be an irrotational two-dimensional motion of an incompressible a inviscid fluid in the xy plane. Let there be no rigid boundaries and let the complex potential of the flow be f(z), where the singularities of f(z) are all at a distance greater than 'a' from the origin. If a circular cylinder of cross section $|z| = a$ is introduced into the fluid and held fixed then the complex potential of the new motion becomes.

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) \tag{4.27}$$

(Let f(z) be a function of a complex variable then its conjugate is $\overline{f(z)}$.

Now, f(z) = u(z) + iv(z)

$$\Rightarrow \bar{f}(z) = u(z) - iv(z)$$

$$\Rightarrow f(\bar{z}) = u(\bar{z}) + iv(\bar{z})$$

$$\Rightarrow \bar{f}(\bar{z}) = u(\bar{z}) - iv(\bar{z})$$

from the above expressions we see that

$$\overline{f(z)} = \bar{f}(\bar{z})$$

This will help in understanding the proof.

Proof : All the singularities of f(z) are in the region $|z|>a$, hence all the singularities of f (a^2/z) lie in the region $|z|<a$. Hence the singularities of $\bar{f}(a^2/z)$ also lie in $|z|<a$. Thus f(z) and $f(z) + \bar{f}(a^2/z)$ both have the same singularities in the region $|z|>a$ and so both functions describe the same hydrodynamical distribution in the region $|z|>a$.

Moreover, on the cylinder $|z| =a$, we take $z = ae^{i\theta}$ and so

$$w= f(z) + \bar{f}(a^2/z)= f(ae^{i\theta})+ \bar{f}(ae^{i\theta})$$

$$= f\left(ae^{i\theta}\right) + \overline{f\left(ae^{i\theta}\right)}$$

Thus on the circle $|z|=a$, w is the sum of a complex quantity and its complex conjugate and is therefore a real number. Hence ψ = Im(z) = 0 on $|z| = a$. This shows that the circular boundary is a stream line across which no fluid flows. Hence $|z| = a$ is a possible boundary for the new flow and $w= f(z) + \bar{f}(a^2/z)$ is the appropriate complex velocity potential for the new flow.

4.6 Flows Represented by Functions of a complex variable

It is well known that if w= f(z) is an analytic function then w is the complex velocity potential of some inviscid incompressible irrotational motion. The real and imaginary parts of this analytic function w represent the velocity potential and the stream function of the motion. We shall now take some analytic functions and discuss the nature of the flow represented by them.

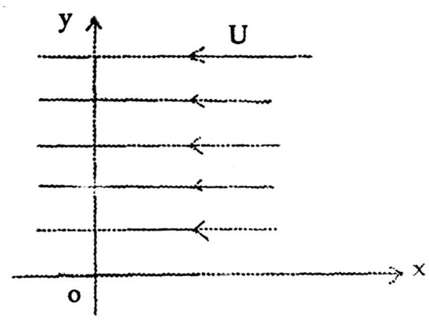
(i) $w = Uz \ (U>0)$ where U is non-zero constant

$$\Rightarrow \phi + i\psi = U(x+iy)$$

$$\Rightarrow \phi = Ux \text{ and } \psi = Uy$$

$$\Rightarrow -\frac{\partial \phi}{\partial x} = -U, \ -\frac{\partial \phi}{\partial y} = 0$$

$$\Rightarrow u \equiv U, \quad v = 0$$



Further, ψ = constant $\Rightarrow$ y = constant

$$w = Uz \qquad \Rightarrow \left|\frac{dw}{dx}\right| = U$$

Hence w=Uz being an analytic function, represents an inviscid incompressible irrotational motion and the motion is uniform with a velocity U parallel to the negative direction of the x- axis.

(ii) $w = Uze^{-i\alpha}$

$$\Rightarrow \phi + i\psi = U(x+iy)(\cos\alpha - i\sin\alpha)$$

$$\Rightarrow \phi = U(x\cos\alpha + y\sin\alpha)$$

And $\psi = U(y\cos\alpha - x\sin\alpha)$

$$-\frac{\partial \phi}{\partial x} = -U\cos\alpha, \quad -\frac{\partial \phi}{\partial y} = -U\sin\alpha$$

ψ = constant $\Rightarrow$ y = tanα x + constant

$$\left|\frac{dw}{dz}\right| = U$$

Hence the complex potential w = Uze$^{-i\alpha}$ represents an inviscid incompressible irrotational uniform motion in which stream line are inclined at an angle α with x- axis. The magnitude of the flow is U.

(iii) $w = \dfrac{Ua^2}{z}$

$$\Rightarrow w = \frac{Ua^2}{re^{i\theta}} = \frac{Ua^2}{r}\left(\cos\theta - i\sin\theta\right)$$

$$\Rightarrow \phi = \frac{Ua^2}{\sqrt{x^2+y^2}}\cos\theta = \frac{-Ua^2 y}{\sqrt{x^2+y^2}}\sin\theta$$

$$\Rightarrow \phi = \frac{Ua^2 x}{x^2+y^2}, \qquad \psi = \frac{-Ua^2 y}{x^2+y^2}$$

ψ = constant

$$\Rightarrow x^2 + y^2 - 2Ay = 0$$

The stream lines ψ = constant are circles of radius a touching the x- axis at origin.

The value of $\left(-\dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}\right)_{r=a} = U\cos\theta$

The normal velocity of the fluid at r = a is U$\cos\theta$. Now if the circle moves with velocity U in the x direction then the normal velocity at r=a will be U$\cos\theta$.

Hence $W = \dfrac{Ua^2}{z}$ represents the complex potential for the flow produced by the motion of a circular cylinder in the direction of x.

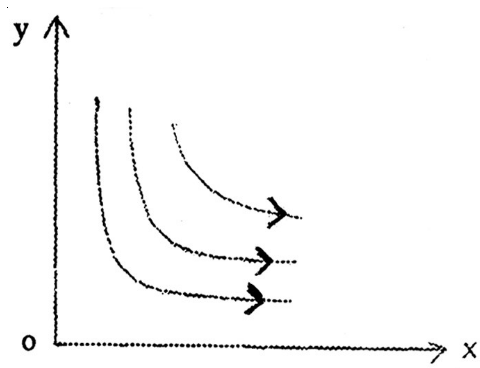
(iv) $w = U\left(z + \frac{a^2}{z}\right)$

We know that the complex potential w = Uz represents a uniform flow with velocity U parallel to x axis in the direction of x-axis of an inviscid incompressible fluid. Now if a cylinder $|z| = a$ is introduced into the flow and held fixed then the new complex potential for the modified flow (by using Milne-Thomson circle theorem) will be given by $w = Uz + U(a^2/z)$ which is same as the complex potential given in the problem. Hence we can say that the complex potential $w = U(z + a^2/z)$ represents the flow in presence of a cylinder $|z| = a$. The velocity at a large distance from the cylinder is parallel to x -axis and is of magnitude U.

(v) $w = Az^2$

$w = Az^2$ gives $\phi = A(x^2 - y^2)$, $\psi = 2Axy$

In this case stream lines are rectangular hyprebolas having the axes of reference as asymtotes. Any two stream lines can be replaced by solid boundaries and this will represent the motion between them. If axes of reference becomes the solid boundary lines, this stream function and velocity potential give the flow round a rectrangular corner.



4.7 Blasius Theorem

An incompressible fluid moves steadily and irrotationally under no external forces parallel to the z-plane past a fixed cylinder whose cross section in that plane is bounded by a closed curve C. The complex potential for the flow is w. then the action of the fluid pressure on the cylinder is equivalent to a force per unit length having components [X, Y] and a couple per unit length of moment M, where

$$Y + iX = -\frac{\rho}{2}\int_C \left(\frac{dw}{dz}\right)^2 dz \; ; \quad M = \text{Re}\left\{-\frac{\rho}{2}\int_C z\left(\frac{dw}{dz}\right)^2\right\} \qquad (4.28)$$

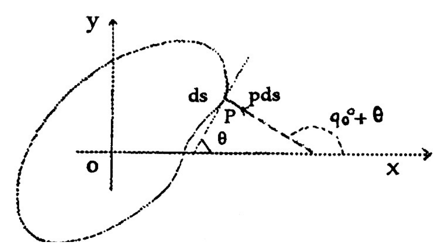
Proof : Consider and element of arc ds surrounding if a point P(x,y) of the fixed cylinder. The tangent at P makes an angle θ with the x-axis. The thrust at P acts inward along the normal at P, its components being

$$pds\ \cos\left(90^0 + \theta\right), \quad pds\ \sin\left(90^0 + \theta\right)$$

i.e. $\quad -pds\ \sin\theta, \quad pds\ \cos\theta$

Hence $X = \int_c -pds\ \sin\theta = \int_c -pdy$

$$Y = \int_c pds\ \cos\theta = \int_c pdx$$



Moment of the thrust pds about O is

$$M = \int_c \left[-\left(-pds\ \sin\theta . y\right) + \left(pds\ \cos\theta . x\right)\right]$$

$$= \int_c p\left(xdx + ydy\right)$$

$$= \int_c p\ \text{Re}\left(zd\bar{z}\right) \qquad (4.29)$$

Also,

$$Y + iX = \int_c p\left(dx - idy\right)$$

$$= \int_c pd\bar{z} \qquad (4.30)$$

In steady flow, from-Bernoulli's equation

$$P + \frac{\rho}{2}\left(u^2 + v^2\right) = P_0 \qquad (4.31)$$

and from the square of the magnitude of the complex conjugate velocity

$$\left|\frac{dw}{dz}\right|^2 = \frac{dw}{dz}.\frac{d\overline{w}}{d\overline{z}} = u^2 + v^2$$

it follows that $p = p_0 \frac{\rho}{2}\frac{dw}{dz}\frac{d\overline{w}}{d\overline{z}}$ (4.32)

Therefore, we write for the force

$$Y + iX = \int_c \frac{\rho}{2}\frac{dw}{dz}\, d\overline{w} \qquad (4.33)$$

because the closed integral over the constant pressure p_0 vanishes. Since the contour of the body is a curve ψ=constant, we have

$$d\overline{w} = d\phi = dw\,,$$

and hence from (4.33) we get

$$Y + iX = -\frac{\rho}{2}\int_c z\left(\frac{dw}{dz}\right)^2 dz \qquad (4.34)$$

In a similar manner, we obtain from (4.29)

$$M = R_e\left\{-\frac{\rho}{2}\int_c z\left(\frac{dw}{dz}\right)^2 dz\right\} \qquad (4.35)$$

Note : The Integration can also be carried out along any arbitrary closed curve enclosing the body, as long as there are no singularities between the contour of the body and the integration curve.

4.8 Motion of Cylinders

Consider a two dimensional irrotational motion produced by motion of a cylinder of radius a in an infinite mass of liquid which is at rest at infinity. For simplicity, we suppose the cylinder to be of unit length and the liquid and the cylinder to be confined between two smooth parallel planes at right angles to the axis of the cylinder.

Since the motion is irrotational, the velocity potential ϕ exists and satisfies Laplaces equation

$$\nabla^2\phi = 0, \text{ i.e. } \frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0 \text{ everywhere} \quad (4.36)$$

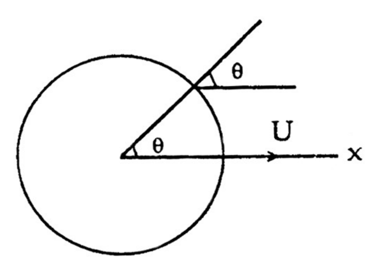
Also since the liquid is at rest at infinity, the velocity components

$$-\frac{\partial\phi}{\partial r}=0, -\frac{1}{r}\frac{\partial\phi}{\partial r}=0 \text{ at infinity} \quad (4.37)$$

At the boundary of the moving cylinder, the normal component of the velocity of the liquid must be equal to the normal component of the velocity of the cylinder.

i.e. $$-\left.\frac{\partial\phi}{\partial r}\right|_{r=a}=U\cos\theta \quad (4.38)$$

where U is the velocity of the circular cylinder in the direction of the x-axis.



The boundary condition (4.38) suggests that ϕ must be of the form $F(r)\cos\theta$.

On this ground we assume that

$$\phi = F(r)\cos\theta \tag{4.39}$$

is a solution of (4.36). Substituting this form of ϕ in (4.36), we get

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} - \frac{F}{r^2} = 0 \tag{4.40}$$

Assuming $F(r)=Ar^m$ above equation gives

$$m(m-1) + m-1 = 0$$

$$\therefore\ m=1, -1$$

Hence, $$\phi(r,\ \theta) = \left(Ar + \frac{B}{r}\right)\cos\theta \tag{4.41}$$

using boundary conditions (4.37) and (4.38) we get

$$\phi = \frac{Ua^2}{r}\cos\theta \tag{4.42}$$

This gives $$\psi = -\frac{Ua^2}{r}\sin\theta \tag{4.43}$$

And hence $w(z) = (\phi + i\psi) = \dfrac{Ua^2}{r}$

In the case when the cylinder is at rest and fluid is moving past the cylinder we can obtain the solution of (4.36) in a similar way as

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta \tag{4.44}$$

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta \tag{4.45}$$

$$w = U\left(z + \frac{a^2}{z}\right) \tag{4.46}$$

4.8.1 Circulation about a Fixed Circular Cylinder :

The space occupied by liquid in presence of a circular cylinder of radius a is a doubly connected region, hence a cyclic motion is possible. Let ϕ be the velocity potential for a cyclic motion with a constant circulation K about the cylinder the cylinder, then for any radius r, we have

$$\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right).2\pi r = K \tag{4.47}$$

where $\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$ is the velocity in the direction of θ. On integration of (4.47), we get

$$\phi = -\frac{K\theta}{2\pi} \tag{4.48}$$

the constant of integration is taken to be zero, since there is no loss of generality in it. the conjugate complex of ϕ gives ψ as

$$\psi = \frac{K}{2\pi}\log\, r \tag{4.49}$$

and hence the complex potential is given by

$$w(z) = \phi + i\psi$$

or $$w(z) = \frac{ik}{2\pi}\log\, z \tag{4.50}$$

4.8.2 Forces on a fixed circular cylinder with circulation

We have seen in section 4.8 that the complex potential in the case when a circular cylinder of radius a is fixed in a stream of uniform velocity U is given by

$$Uz + \frac{Ua^2}{z}$$

and in section 4.8.1 that the complex potential due to a circulation K about the cylinder is given by

$$\frac{iK}{2\pi}\log z$$

Hence the complex potential w(z), for the combined motion is given by

$$w(z) = U\left(z + \frac{a^2}{z}\right) + \frac{iK}{2\pi}\log z \qquad (4.51)$$

Now, $\dfrac{dw}{dz} = U\left(1 - \dfrac{a^2}{z^2}\right) + \dfrac{iK}{2\pi z}$

$$\left(\frac{dw}{dz}\right)^2 = U^2\left(1 - \frac{a^2}{z^2}\right)^2 - \frac{K^2}{4\pi^2 z^2} + \frac{iKU}{\pi z}\left(1 - \frac{a^2}{z^2}\right) = F(z), \text{ say}$$

If X,Y be the components of thrust on the cylinder, then,by Blasius theorem , we have

$$X - iY = \frac{1}{2} i\rho \int_{|z|=a} \left(\frac{dw}{dz}\right)^2 dz \qquad (4.52)$$

the poles of the function F(z) inside |z|=a is at z=0 and the residue there is $\dfrac{iKU}{\pi}$

Hence, by Cauchy's residue theorem, we have

$$\int_{|z|=a} \left(\frac{dw}{dz}\right)^2 dz = \int_{|z|=a} F(z)\,dz = 2\pi i.\frac{iKU}{\pi} = -2KU$$

Therefore, equation (3.52) gives

$$X - iY = \frac{1}{2} i\rho(-2KU)$$

$$\Rightarrow x = 0, \quad \text{and} \quad y = \rho \text{KU} \qquad (4.53)$$

which means that the cylinder experiences and upward lift. This lifting effect produced by the circulation is called the Magnus effect.

When circulation is not present i.e. K = 0, the force on the cylinder is zero. This means that when a circular cylinder is fixed in uniform stream it does not experience any resistance, but experiments show that it is not so. This is known as D$^{/}$. Alembert Paradox. The character of the motion in the neighbourhood of the cylinder is completely changed when even a small amount of viscosity is present in the fluid and then the cylinder experiences resistence.

4.9 Check your progress

Q.1. What do you mean by two dimensional flow?

Q.2. What is stream function? Is it a harmonic function?

Q.3. State the physical significance of steam function.

Q.4. State the Milne - Thomeson Circle theorem?

Q.5. What is Blasius Theorem?

Q.6 Prove that the two-dimensional irrotational motion of a liquid bounded by the lines y=0, y=2a due to a source at the point (0,a) is given by the complex velocity potential

$$\text{W} = -m \log \cosh\left(\frac{\pi z}{2a}\right)$$

where m is the strength of the source.

Q.7. Verify that $\text{W} = i\text{K}\log\left(\frac{z-ia}{z+ia}\right)$ is the complex potential of a steady flow of a liquid about a circular cylinder, the plane y=0 being a rigid boundary. Find the force exerted by the liquid on the unit length of the cylinder

$$W = iK \log\left(\frac{z-ia}{z+ia}\right)$$

Q.8. In a two-dimensional irrotational motion of an incompressible inviscid fluid, a source of strength m is placed at each of the points (-1, 0) and (1,0) and a sink of strength 2m is placed at the origin. Show that the stream lines are

$$\left(x^2+y^2\right)^2 = x^2 - y^2 + Kxy$$

where K is a parameter.

Q.9. Show that the force exerted on a cylinder $|z| = a$ in the irrotational flow produced by a line source of strength is $X = pm^2/(48\pi a), Y = 0$

Q.10 A sphere of radius a is moving with constant velocity U through an infinite liquid at rest at infinity. If p_∞ be the pressure at infinity, show that the pressure at any point on the surface of the sphere, the radius through which point makes an angle θ with the direction of motion is given by

$$P = P_\infty + \tfrac{1}{2}\rho U^2\left(1 - \tfrac{9}{4}\sin^2\theta\right)$$

4.10 Let us sum up

1. If a fluid moves in such a way that at any given instant the flow pattern in a certain plane is the same as that in all other parallel planes within the fluid then the flow is said to be two dimensional.
2. In irrotational two-dimensional motion the velocity potential and the stream function, both satisfy Laplace equation.

3. Stream function is constant along a stream line.
4. An inviscid incompressibe irrotational flow is governed by Laplace equation.
5. When a circular cylinder is fixed in a uniform stream it does not experience any resistance, but experiment show that it is not so. This is known as D' Alemberts Paradox. The character of motion in the neighbourhood of the cylinder is completely changed when even a small amount of viscosity is present in the fluid and then the cylinder experiences resistance.
6. The lift around an aerofoil is generated when the flow possesses circulation. The lift around a body of any shape is given by $K\rho U$ where ρ is the density, K is the circulation and U is the velocity in the stream wise direction.