SELF LEARNING MATERIAL

Master of Arts/Science

MATHEMATICS

COURSE: MATH - 202

TENSOR

BLOCK : 1, 2, 3, 4 & 5

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MATHEMATICS

COURSE : MATH - 202

TENSOR

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MATHEMATICS

COURSE : MATH - 202

TENSOR

BLOCK - 1

BASIC CONCEPTS

List of Contents :

1.0 : Objectives

After working with this block you will be able to

- Use summation convention to write long mathematical expressions in short.
- Define Kronecker deltas and permutation symbols
- Write determinants in tensor notations
- Understand about curvilinear coordinates
- Write a vector by using different basis.

1.1 Introduction:

In our discussion here, we have included : summation convention, definitions and properties of Kronecker delta, permutation symbols, determinants and curvilinear coordinates. We have discussed certain examples to make you familiar with the methods of solving problems related to this unit. We have also suggested interesting activities that you may attempt as we go along.

1.2 Subscripts and Superscripts:

Let us consider three mutually orthogonal straight lines OX^1, OX^2, and OX^3 in the right handed orientation. These lines can determine uniquely the position of a point, and such lines can be taken as coordinates axes with O as origin in an Euclidean space of three dimensions. It is often convenient to denote the coordinate with respect to these axes by x^1, x^2, x^3 instead of x, y, z. Thus, we refer to the coordinates of a point x^1, x^2, x^3 as the point x^i, where i takes the values 1, 2, 3. It may be noted here that the numbers 1, 2, 3 written above x are not powers but are used merely to distinguish the variables.

Following this notation, we can write the equation of a plane in the form

$$a_1 x^1 + a_2 x^2 + a_3 x^3 + a = 0$$

where a's are constants.

The suffixes i and j in A^i_j are called superscript and subscript respectively. The upper position always denotes superscript and the lower position denotes subscript. Superscripts must not be confused with exponents. If doubt arises, an exponent may be distinguished from a superscript by using a bracket. Thus the square of x^i may be denoted by $(x^i)^2$.

Indices play an important role in 'Tensor Analysis'. These indices may range over from 1 to any finite natural number n but the physical meaningful range of the values of the indices is when $n \leq 3$. We shall, unless otherwise stated, restrict the range of the values of the indices to 1, 2, 3 only. Any index occurring only once in a given term is called a free index. Note that any free index which appears must appear in the same position in each term of an equation.

1.3 Summation convention:

Any term in which the same index appears twice, once as a superscript and once as a subscript is known as a dummy index and stands for the sum of all such terms obtained by giving this index its entire range of values. This is known as summation convention (introduced by Einstein).

For illustration, we can write

$$a_1^1 + a_2^2 + a_3^3 \quad \text{as} \quad a_i^i,$$

$$a^1 b_1 + a^2 b_2 + a^3 b_3 \quad \text{as} \quad a^i b_i,$$

$$a_1^i x^1 + a_2^i x^2 + a_3^i x^3 \quad \text{as} \quad a_j^i x^j$$

In the last example, the term $a_j^i x^j$ represents three expressions for $i = 1, 2, 3$. A term may have more than one index repeated. Then all repeated indices are to be summed over, as the following:

$$\begin{aligned} a_{ij}x^i x^j = {} & a_{11}x^1x^1 + a_{12}x^1x^2 + a_{13}x^1x^3 \\ & + a_{21}x^2x^1 + a_{22}x^2x^2 + a_{23}x^2x^3 \\ & + a_{31}x^3x^1 + a_{32}x^3x^2 + a_{33}x^3x^3 \end{aligned}$$

Note that whether the summaton on i is carried out first or on j, does not matter. It is important to realize that i and j are dummy indices and may be replaced by any other distinct letters; i.e.

$$a_{ij}x^i x^j = a_{pq}x^p x^q = a_{\alpha\beta}x^\alpha x^\beta$$

we would never write $a^{ij}b_{jj}$ because the nature of the summations in such an expression is not well defined; that is, we must denote different index summations by different letters.

Note also that the summaton convention does not apply to numerical indices. For instance a_3x^3 stands for a single term.

A superscript in the denominator of a term is regarded as a subscript. Thus,

$$\frac{\partial v^i}{\partial x^i} = \frac{\partial v^1}{\partial x^1} + \frac{\partial v^2}{\partial x^2} + \frac{\partial v^3}{\partial x^3}.$$

Example 1.1

If ϕ is a function of x^1, x^2, x^3 prove that $d\phi = \frac{\partial \phi}{\partial x^i} dx^i$.

Solution : Since ϕ is a function of x^1, x^2, x^3,

therefore,

$$d\phi = \frac{\partial \phi}{\partial x^1} dx^1 + \frac{\partial \phi}{\partial x^2} dx^2 + \frac{\partial \phi}{\partial x^3} dx^3$$

$$\Rightarrow d\phi = \frac{\partial \phi}{\partial x^i} dx^i$$

Activity 1.1

1. Write the following using summation convention:

(i) $a_1x^1x^3 + a_2x^2x^3 + a_3x^3x^3$

(ii) $A^{21}B_1 + A^{22}B_2 + A^{23}B_3$

(iii) $g^{21}g_{11} + g^{22}g_{21} + g^{23}g_{31}$

2. Expand the following using the summation convention:

(i) $A_{1q} B^{q2}$, (ii) $\frac{\partial}{\partial x^k}\left(\sqrt{g}\, A^k\right)$

After studying summation convention, please answer the questions below. If you are able to answer all the questions, you can move on to the next section. However, if you are not able to solve all the questions, you need to revisit this section. Answer to these questions are given at the end of this unit.

Ex. 1.1 Detect the mistake if we write for $i = 1, 2, 3, \ldots, n$

$x_i x^i = x_1 x^1 + x_2 x^2 + \ldots + x_n x^n.$

Ex. 1.2 State 'true' or 'false'

$a_{ij}\, x^i = a_{1j}\, x^1 + a_{2j}\, x^2 + a_{3j}\, x^3$

Ex. 1.3 How many expressions are represented by (i) $a_i x^i$, (ii) $a_i x^j$.

1.4 Kronecker delta, Permutation symbols and Generalized Kronecker delta.

*** Kronecker delta :** The Kronecker delta is defined as

$$\delta^i_j = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases} \tag{1.1}$$

In some cases δ^i_j is also written as δ_{ij} or δ^{ij}.

Thus we have

$$x^i = \delta^i_j x^j, \qquad \delta_{ij} x^i x^j = x^1.x^1 + x^2.x^2 + x^3.x^3$$

$$\delta^i_i = \delta^1_1 + \delta^2_2 + \delta^3_3 = 1 + 1 + 1 = 3,$$

$$\delta^p_l \delta^r_p B^l = \delta^r_l B^l = B^r.$$

Now try to answer the following questions

Ex. 1.4 Simplify $\delta^i_j \delta^k_l B^{jl}$

Ex. 1.5 Show that $\dfrac{\partial x^m}{\partial x^n} = \delta^m_n$

Ex. 1.6 Show that $\dfrac{\partial x^m}{\partial u^p}\dfrac{\partial u^p}{\partial x^n} = \delta^m_n$

*** Permutation symbols :** The permutation symbols e_{ijk} and e^{ijk} are defined by

$$e_{ijk} = e^{ijk} = \begin{cases} +1 & \text{if } i, j, k \text{ form an even permutation of } 1,2,3 \\ -1 & \text{if } i, j, k \text{ form an odd permutation of } 1,2,3 \\ 0 & \text{if two or more indices are equal} \end{cases}$$

If follows that

$$e^{ijk} = e^{jki} = e^{kij}, \text{ and } e_{ijk} = e_{jki} = e_{kij} \tag{1.2}$$

*** Generalized Kronecker delta :** The generalized Kronecker delta, denoted by δ^{ijk}_{lmn} is defined by

$$\delta^{ijk}_{lmn} = e^{ijk} e_{lmn} \tag{1.3}$$

That is, δ^{ijk}_{lmn} is the product of both the permutation symbols and its value depends on the values of the permutaion symbols. Hence it will also take the values 1, -1, 0. It follows that

$$\delta^{ijk}_{lmn} = \delta^{jki}_{lmn} = \delta^{jki}_{mnl} \tag{1.4}$$

Consider the result of δ^{ijk}_{lmk}. We have

$$\delta^{ijk}_{lmk} = \delta^{ij1}_{lm1} + \delta^{ij2}_{lm2} + \delta^{ij3}_{lm3} = \delta^{ij}_{lm}$$

Thus,

$$\delta^{ij}_{lm} = \begin{cases} +1 & if\ i \neq j\ and\ ij\ are\ even\ permutations\ of\ lm \\ -1 & if\ i \neq j\ and\ ij\ are\ odd\ permutations\ of\ lm \\ 0 & if\ i = j\ or\ l = m \end{cases} \tag{1.5}$$

Also we have

$$\delta^{ijk}_{ljk} = \delta^{ij}_{lj} = \delta^{i1}_{l1} + \delta^{i2}_{l2} + \delta^{i3}_{l3} = 2\delta^{i}_{l} \tag{1.6}$$

Theoem (1.1) : δ^{ij}_{lm} is related to the Kronecker delta as

$$\delta^{ij}_{lm} = \delta^{i}_{l}\delta^{j}_{m} - \delta^{i}_{m}\delta^{j}_{l} = \begin{vmatrix} \delta^{i}_{l} & \delta^{i}_{m} \\ \delta^{j}_{l} & \delta^{j}_{m} \end{vmatrix} \tag{1.7}$$

Proof: The principle of conservation of indices requires that any free index, which appears must appear in the same position in each term of an equation.

By this principle, we have

$$\delta^{ij}_{lm} = \alpha\,\delta^{i}_{l}\delta^{j}_{m} + \beta\,\delta^{i}_{m}\delta^{i}_{l}$$

where α, β are unkown constants. Since above relation is an indentity, we have

$$\delta^{12}_{12} = \alpha\,\delta^{1}_{1}\delta^{2}_{2} + \beta\,\delta^{1}_{2}\delta^{2}_{1} \Rightarrow 1 = \alpha$$

$$\delta^{12}_{21} = \alpha\,\delta^{1}_{2}\delta^{2}_{1} + \beta\,\delta^{1}_{1}\delta^{2}_{2} \Rightarrow -1 = \beta$$

Hence,

$$\delta^{ij}_{lm} = \delta^{i}_{l}\delta^{j}_{m} - \delta^{i}_{m}\delta^{j}_{l}$$

Theorem (1.2): δ^{ijk}_{lmn} , δ^{ij}_{lm} , δ^{i}_{l} etc. are related as

$$\delta^{ijk}_{lmn} = \delta^{i}_{l}\delta^{jk}_{mn} + \delta^{i}_{m}\delta^{jk}_{nl} + \delta^{i}_{n}\delta^{jk}_{lm} \tag{1.8}$$

$$= \begin{vmatrix} \delta^{i}_{l} & \delta^{i}_{m} & \delta^{i}_{n} \\ \delta^{j}_{l} & \delta^{j}_{m} & \delta^{j}_{n} \\ \delta^{k}_{l} & \delta^{k}_{m} & \delta^{k}_{n} \end{vmatrix} \tag{1.9}$$

Proof : By the principle of conservation of indices we have

$$\delta^{ijk}_{lmn} = \alpha\,\delta^{i}_{l}\,\delta^{jk}_{mn} + \beta\,\delta^{i}_{m}\,\delta^{jk}_{nl} + \gamma\,\delta^{i}_{n}\,\delta^{jk}_{lm}$$

where α, β, ν are unknown constants. By using the fact that above relaton is an identity, we get

$$\delta^{123}_{123} = \alpha\, \delta^{1}_{1}\delta^{23}_{23} + \beta\, \delta^{1}_{2}\, \delta^{23}_{31} + \nu\, \delta^{1}_{3}\delta^{23}_{12} \Rightarrow 1 = \alpha$$
$$\delta^{123}_{231} = \alpha\, \delta^{1}_{2}\delta^{23}_{31} + \beta\, \delta^{1}_{3}\, \delta^{23}_{21} + \nu\, \delta^{1}_{1}\delta^{23}_{23} \Rightarrow 1 = \nu$$
$$\delta^{123}_{312} = \alpha\, \delta^{1}_{3}\delta^{23}_{12} + \beta\, \delta^{1}_{1}\, \delta^{23}_{23} + \nu\, \delta^{1}_{2}\delta^{23}_{31} \Rightarrow 1 = \beta$$

Hence we get

$$\delta^{ijk}_{lmn} = \delta^{i}_{l}\delta^{jk}_{mn} + \delta^{i}_{m}\delta^{jk}_{nl} + \delta^{i}_{n}\delta^{jk}_{lm}\,.$$

Using (1.7) we can get (1.9).

Activity 1.2

By using identities (1.7) and (1.9) prove that

(i) $\delta^{ij}_{ij} = \delta^{ijk}_{ijk} = 3! = 6$

(ii) $\delta^{i}_{i} = \dfrac{1}{2}\delta^{ij}_{ij} = \dfrac{1}{2}\delta^{ijk}_{ijk} = 3$

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After studying this section please answer the questions below. If you are able to answer all the questions, you can move on to the next section. However if you are not able to answer all the questions, you need to revisit this section.

Ex.1.7: If a_{ij} $(i, j = 1, 2, 3, \ldots, n)$ are constant quantities such that $a_{ij}\, x^i x^j = 0$ then show that $a_{kl} + a_{lk} = 0$ for all integral values of $l, k = 1, 2, 3, \ldots, n$.

Ex. 1.8: Prove that

(i) $e^{ijk} e_{imn} = \delta^{j}_{m}\delta^{k}_{n} - \delta^{j}_{n}\delta^{k}_{m}$

(ii) $e_{ijk} = \delta^{m}_{i}\, \delta^{n}_{j}\, \delta^{p}_{k}\, e_{mnp}$

1.5 Determinants in tensor notation

Let a_i^j be an element of a determinant occurring in i^{th} row and j^{th} column and a is the value of the determinant. Thus we write

$$a = \det a_i^j = \left|a_i^j\right| = \begin{vmatrix} a_1^1 & a_1^2 & a_1^3 \\ a_2^1 & a_2^2 & a_2^3 \\ a_3^1 & a_3^2 & a_3^3 \end{vmatrix} \qquad \ldots (1.10)$$

The subscripts indicate rows and the superscripts indicate columns. The numerical value a is obtained as follows:

$$a = \left|a_i^j\right| = e_{ijk} a_i^i a_2^j a_3^k = e^{lmn} a_l^1 a_m^2 a_n^3 \qquad \ldots (1.11)$$

The summation on i, j, k or l, m, n gives $3^3 = 27$ terms of which 21 terms involve repeteated indices and therefore are zero whereas the non-vanishing six terms are the same as those in (1.10) which we get by expanding the determinant.

The theorems concerning the interchange of rows and columns are given as

$$a\, e_{pqr} = e_{ijk}\, a_p^i\, a_q^j\, a_r^k \qquad (1.12)$$

$$a\, e^{pqr} = e^{ijk}\, a_i^p\, a_j^q\, a_k^r \qquad (1.13)$$

when p, q, r are 1, 2, 3 then the equations (1.12) and (1.13) reduce to (1.11). When two adjacent indices are interchanged e_{pqr} changes its sign, hence we get

$$e_{ijk} a_p^i a_q^j a_r^k = e_{ijk} a_q^j a_p^i a_r^k = -e_{jik} a_q^j a_p^i a_r^k$$

The cofactor of an element a_i^j in a determinant is defined as its coefficient in the expansion of the determinant and is denoted by A_j^i. Thus we have the relation

$$a_J^i A_K^J = a\, \delta^i{}_K = a_K^J A_J^i \qquad (1.14)$$

It the elements a_i^j of the determinant $a = |a_i^j|$ are functions of x, then the derivative of a with respect to x is given as follows:

$$\frac{da}{dx} = \frac{d}{dx}\left(e^{ijk}\, a_i^1\, a_j^2\, a_k^3\right)$$

$$= e^{ijk}\left[\frac{da_i^1}{dx} a_j^2 a_k^3 + \frac{da_j^2}{dx} a_i^1 a_k^3 + \frac{da_k^3}{dx} a_i^1 a_j^2\right]$$

$$= \frac{da_i^1}{dx} A_1^i + \frac{da_j^2}{dx} A_2^j + \frac{da_k^3}{dx} A_3^k$$

$$\frac{da}{dx} = \frac{da_j^i}{dx} A_i^j \qquad \ldots (1.15)$$

Thus, the derivative of a determinant is the sum of the product of the derivative of each element and the cofactor of that element.

Let us now consider the product of two determenants

$$|a_i^j|\,|b_i^j| = |c_i^j|$$

In this case $\quad c_i^j = a_\alpha^j b_i^\alpha$

1.6 Curvilinear Coordinates:

Let (x, y, z) be the carlesian coordinates of any point P in three dimensional space. Now, we assume that these coordinates can be expressed in terms of three independent single valued continuous differentiable scalar point functions u_1, u_2, u_3 such that

$$X = X(u_1, u_2, u_3), \qquad Y = Y(u_1, u_2, u_3),\ Z = Z(u_1, u_2, u_3) \qquad \ldots (1.16)$$

It is also assumed that the functions possess continuous partial derivative of r^{th} order. Then these functions can be solved in terms of x, y, z that is

The surfaces $u_1 = C_1$, $u_2 = C_2$, $u_3 = C_3$ where C_1, C_2, C_3 are constant, are the respective level surfaces of three functions. It is assumed that these three level surfaces do not coincide or intersect in a common curve. So, for each set of values that may be assigned to C_1, C_2, C_3, there is just one point P at which the three level surfaces meet i.e., a unique point is defined for a set of values given to u_1, u_2, u_3. Then (u_1, u_2, u_3) may be used as coordinates in place of (x, y, z) to level points of space. They are called curvilinear coordinates (since the coordinate lines are curved).

These level surfaces are called as coordinate surfaces, through the point P and their three curves of intersection are called coordinate curves and the tangents at P to the coordinate curves are the coordinates axes. The directions of the axes vary from point to point. Along u_1 - coordinate line, the other two parameters u_2, u_3 remain constant and u_1 only varies. Similar is the case for other two coordinate lines.

It is worth noting that a condition that the three level surfaces of the u's through any point P meet at no other point near P is that the normals to the level surfaces are non-coplanar at P.

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then on substituting x, y, z from equation (1.16) we have $\vec{r} = \vec{r}\ (U_1, U_2, U_3)$.

Thus the differential at the point P is given by

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 \qquad (1.18)$$

If u_2 and u_3 are kept constant, so that $du_2 = du_3 = 0$ and if $du > 0$ then the differential $d\vec{r}$ is the direction of the tangent to the coordinate line (in the sense of u_1 increasing). Therefore, the tangent to the u_1 - coordinate curve is parallel to $\frac{d\vec{r}}{\partial u_1}$. Similar results apply for the other coordinate curves, and so if $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are tangent vectors not necesarily of unit length in the directions of u_1, u_2, u_3 increasing respective, then

$$\frac{\partial \vec{r}}{\partial u_1} = \vec{e}_1, \quad \frac{\partial \vec{r}}{\partial u_2} = \vec{e}_2, \quad \frac{\partial \vec{r}}{\partial u_3} = \vec{e}_3 \qquad (1.19)$$

These $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are known as fundamental base vectors.

On taking magnitudes in (1.19), we have

$$\left|\vec{e}_1\right| = h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|, \left|\vec{e}_2\right| = h_2 = \left|\frac{\partial \vec{r}}{\partial u_2}\right|, \left|\vec{e}_3\right| = h_3 = \left|\frac{\partial \vec{r}}{\partial u_3}\right| \qquad (1.20)$$

On substituting from (1.19) in (1.18) we have

$$d\vec{r} = du_1\,\vec{e}_1 + du_2\,\vec{e}_2 + du_3\,\vec{e}_3 \qquad (1.21)$$

When using curvilinear coordinates, it is advisable to introduce, along with the fundamental base vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ the reciprocal base vectors $\vec{e}^1$, $\vec{e}^2$, $\vec{e}^3$ connected with the fundamental base vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ by the formulas

$$\vec{e}_n \cdot \vec{e}^k = \delta_n^k \qquad (1.22)$$

where δ_n^k are the Kronecker deltas.

To do this, it is sufficient to put

$$\vec{e}^1 = \frac{\vec{e}_2 \times \vec{e}_3}{g}, \quad \vec{e}^2 = \frac{\vec{e}_3 \times \vec{e}_1}{g}, \quad \vec{e}^3 = \frac{\vec{e}_1 \times \vec{e}_2}{g}, \qquad (1.23)$$

where $g = \vec{e}_1 \cdot \left(\vec{e}_2 \times \vec{e}_3\right) = \left[\vec{e}_1\ \vec{e}_2\ \vec{e}_3\right]$

From (1.22) it also follows that

where $G = \vec{e}^1 \cdot \left(\vec{e}^2 \times \vec{e}^3\right) = \left[\vec{e}^1\ \vec{e}^2\ \vec{e}^3\right]$ (1.24)

Thus $\vec{e}^1$ is perpendicular to the ($\vec{e}^2$, $\vec{e}^3$) plane.

If the coordinate system is orthogonal, it is obvious (see (1.23) and (1.24)) that the base vectors $\vec{e}_n$ and $\vec{e}^n$ coincide in directions, but their magnitudes are in general different.

In case of curvilinear coordinate system u_1, u_2, u_3 ,the vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ tangential to the coordinate curves u_1, u_2, u_3 respectively are taken as the basis vectors. The basis $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ is said to be local since in general it varies from point to point. It should be noted that in general the basis vectors are neither perpendicular to each other nor of unit length. The reciprocal vectors $\vec{e}^1, \vec{e}^2, \vec{e}^3$ can be taken as another basis. So, any vector can be represented in terms of $\vec{e}_1, \vec{e}_2, \vec{e}_3$ as well as $\vec{e}^1, \vec{e}^2, \vec{e}^3$. Thus we see that two sets of basis can be used to represent a vector.

Answers:

Activity 1.1

1.1 (i) $a_i x^i x^3$

(ii) $A^{2i} B_i$

(iii) $g^{2i} g_{i1}$

1.2. (i) $A_{11} B^{12} + A_{12} B^{22} + A_{13} B^{32}$

(ii) $\frac{\partial}{\partial x^1}\left(\sqrt{g}A^1\right)+\frac{\partial}{\partial x^2}\left(\sqrt{g}A^2\right)+\frac{\partial}{\partial x^3}\left(\sqrt{g}A^3\right)$

Answer of Questions:

1.1 $x_i x^i = x_1 x^1 + x_2 x^2 + \ldots +$ to n terms

1.2 True

1.3 (i) 1, (ii) 9

1.4 B^{ik}

1.5 $\frac{\partial x^m}{\partial u^p}\frac{\partial u^p}{\partial x^n} = \frac{\partial x^m}{\partial x^n} = \delta^m_n$

1.7 Let Us Sum Up

In this block you have learnt about subscripts, superscripts and the summation convention using which it becomes possible to write in short the long expressions consisting of sum of similar terms. You have also learnt about Kronecker delta, Permutation symbols, Generalized Kronecker delta and their properties. You have also seen how a determinant can be written in short in tensor notation. You have also been introduced with curvilinear coordinates. In the latter part of this self learning material you will need to use these concepts.

BLOCK - 2

TENSOR ALGEBRA

List of Contents :

2.0 : Objectives

After working with this block you will be able to

- define various types of tensors
- perform various algebraic operations on tensors
- check up whether a given quantity is a tensor or not

2.1 Introduction :

In our discussion here we have included : the definitions of various types of tensors, algebraic operations and their properties, quotient rule and its applications. We have discussed certain examples also to make you familiar with various operations on tensors.

2.2 Scalars, vectors and tensors:

A scalar is a quantity that can be specified in any coordinate system by just one number, whereas the specification of a vector requires three numbers, namely its components with respect to some basis in three dimensional space. Both scalars and vectors are special cases of a more general concept called a tensor (of order n) whose specification in any given coordinate system requires 3^n numbers, called the components of the tensor. More specifically a tensor is defined as a system of quantities or functions whose components obey certain laws of transformation of coordinates from one system to the other. The key proeperty of a tensor is the transformation law of its components. These components are functions of positions.

Physical laws must be independent of any particular coordinate system used in describing them mathematically if they are to be valid.

Since tensors have useful properties that are independent of coordinate system hence they are used to represent various fundamental laws of physics, engineering, science and mathematics.

2.3 Contravariant and Covaraint vectors :

If a set of n functions A^i of the coordinates x^i transforms into a set of n functions $\overline{A}^i$ of the coordinates $\overline{x}^i$ by the law

$$\overline{A}^i = \frac{\partial \overline{x}^i}{\partial x^k} A^k \qquad (2.1)$$

we say that the functions $\overline{A}^i$ are the components of a contravariant vector in $\overline{x}$- system and A^k are the components of a contravariant vector in x - system.

If we multiply relations (2.1) by $\frac{\partial x^s}{\partial \overline{x}^i}$, we get

$$\frac{\partial x^s}{\partial \overline{x}^i} \overline{A}^i = \frac{\partial x^s}{\partial \overline{x}^i} \frac{\partial \overline{x}^i}{\partial x^k} A^k = \delta^s_k = A^s$$

Hence $$A^s = \frac{\partial x^s}{\partial \overline{x}^i} \overline{A}^i \qquad \ldots (2.2)$$

Note : (i) Above relation is called the transformation law of a contravariant vector. Instead of saying that $A^1, A^2, \ldots$ are the components of a contravariant vector, we simply say that A^i is a contravariant vector.

(ii) A contravariant vector is indicated by superscripts.

(iii) By the coordinate transformation we do not get a new vector (tensor) but the components of the same vector (tensor) are changed.

Example 2.1 : Show that the quantities dx^i (the infinitesimal displacement) are contravariant vectors.

Solution : Let $\bar{x}^p = \bar{x}^p(x^1, x^2, \ldots x^N)$ then

$$d\bar{x}^p = \frac{\partial \bar{x}^p}{\partial x^1} dx^1 + \frac{\partial \bar{x}^p}{\partial x^2} dx^2 + \ldots \text{ to } N \text{ terms}$$

$$\Rightarrow d\bar{x}^p = \frac{\partial \bar{x}^p}{\partial x^i} dx^i$$

which shows that the quantities dx^i form contravariant vector.

If a set of n functions A_i of the coordinates x^i transforms into a set of n functions $\bar{A}_i$ of the coordinates $\bar{x}^i$ by the law

$$\bar{A}_i = \frac{\partial x^p}{\partial \bar{x}^i} A_p \qquad (2.3)$$

we say that the functions A_p are the components of a covariant vector in x^i system and $\bar{A}_i$ are the covariant components of the same vector in $\bar{x}^i$ system.

The equation (2.3) can also be put in the form

$$A_p = \frac{\partial x^p}{\partial \bar{x}^i} \bar{A}_i \qquad (2.4)$$

Note : A covariant vector is indicated by subscripts.

Definition 2.1 : If a function ϕ of the coordinates x^i transform into a function $\bar{\phi}$ of the coordinates $\bar{x}^p$, in such a way that $\phi = \bar{\phi}$ then ϕ is called a scalar.

Example 2.2

Let ϕ be a scalar function, that is,

$$\phi(x^i) = \bar{\phi}(\bar{x}^p)$$

Then $\dfrac{\partial \overline{\phi}}{\partial \overline{x}^p} = \dfrac{\partial \phi}{\partial \overline{x}^p} = \dfrac{\partial \phi}{\partial x^i} \dfrac{\partial x^i}{\partial \overline{x}^p}$.

If we denote $\dfrac{\partial \phi}{\partial \overline{x}^p}$ by $\overline{A}_p$ and $\dfrac{\partial \phi}{\partial x^i}$ by A_i then above equation takes the form

$$\overline{A}_p = A_i \frac{\partial x^i}{\partial \overline{x}^p}$$

which shows that A_i i.e. $\dfrac{\partial \phi}{\partial x^i}$ transforms like a covariant vector.

2.4 : Tensors of rank two

If a set of N^2 functions A^{ij} of the coordinates system x^i transforms into a set of N^2 functions $\overline{A}^{pq}$ of another coordinate system $\overline{x}^p$ by the law

$$\overline{A}^{pq} = \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial \overline{x}^q}{\partial x^j} A^{ij} \qquad (2.4.1)$$

we say that the functions A^{ij} are the contravariant components of a tensor of order two in the x - system and $\overline{A}^{pq}$ are the contravariant componets of the same tensor in the $\overline{x}$ - system.

If a set of N^2 functions A^i_j of the coordinate system x^i transforms into a set of N^2 functions $\overline{A}^p_q$ of another coordinate system $\overline{x}^p$ by the law

$$\overline{A}^p_q = \frac{\partial \overline{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \overline{x}^q} A^i_j$$

we say that the functions A^i_j are the components of a mixed tensor of order (1+1) in the x - system and $\overline{A}^p_q$ are the components of the same tensor in the $\overline{x}$ - system.

Exasmple 2.3: Show that Kronecker delta is a mixed tensor of order 2.

Solution : In the $\bar{x}$ - system, by definition

$$\bar{\delta}^p_q = \frac{\partial \bar{x}^p}{\partial \bar{x}^q}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}_q}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}_q} \delta^i_j$$

which is the law of transformation of a mixed tensor of rank two. Hence Kronecker delta is a mixed tensor of rank two.

2.5 : Relative and Absolute tensors

A tensor $A^{p_1 \cdots p_M}_{r_1 \ldots r_N}$ is called a relative tensor of weight ω if its components transform according to the equation

$$\bar{A}^{q_1 \cdots q_M}_{s_1 \ldots s_N} = \left|\frac{\partial x}{\partial \bar{x}}\right|^w \frac{\partial \bar{x}^{q_1}}{\partial x^{p_1}} \cdots \frac{\partial \bar{x}^{q_M}}{\partial x^{p_M}} \cdot \frac{\partial x^{r_1}}{\partial \bar{x}^{s_1}} \cdots \frac{\partial x^{r_N}}{\partial \bar{x}^{s}} A^{p_1 \cdots p_M}_{r_1 \ldots r_N}$$

where $J = \left|\frac{\partial x}{\partial \bar{x}}\right|$ is the Jacobian of the transformation. If $w = 0$ the tensor is called absolute and is the type of tensors which we have defined above. If $w = 1$ the relative tensor is of weight one.

Activity 2.1

Define a relative tensor of order (2 +1) and weight 2.

- -

- -

- -

2.6 : Symmetric and Skew Symmetric Tensors

A tensor is called symmetric with respect to two contravariant or two covariant indices if its components remain unaltered upon interchange of the indices. Thus if $A_{qs}^{npr} = A_{qs}^{pnr}$ the tensor is symmetric in n and p. If a tensor is symmetric with respect to any two contravariant and any two covariant indices, it is called symmetric.

A tensor is called skew-symmetric with respect to two contravariant or two covariant indices if its components change sign upon interchange of the indices. Thus if $A_{pq}^{lmn} = -A_{pq}^{mln}$ the tensor is said to be skew-symmetric in l and m. If a tensor is skew-symmetric with respect to any two contravariant and any two covariant indices it is called skew-symmetric.

2.7 : Operation on Tensors:

Addition : The sum of two or more tensors of the same rank and type is also a tensor of the same rank and type. Let A_k^{ij} and B_k^{ij} be two mixed tensor of order (2+1) then

$$\overline{A}_\nu^{\alpha\beta} = A_k^{ij} \frac{\partial \overline{x}^\alpha}{\partial x^i} \cdot \frac{\partial \overline{x}^\beta}{\partial x^j} \cdot \frac{\partial x^k}{\partial \overline{x}^\nu}$$

$$\overline{B}_\nu^{\alpha\beta} = B_k^{ij} \frac{\partial \overline{x}^\alpha}{\partial x^i} \cdot \frac{\partial \overline{x}^\beta}{\partial x^j} \cdot \frac{\partial x^k}{\partial \overline{x}^\nu}$$

Adding we get

$$\overline{A}_\nu^{\alpha\beta} + \overline{B}_\nu^{\alpha\beta} = \left(A_k^{ij} + B_{jk}^{i}\right) \frac{\partial \overline{x}^\alpha}{\partial x^i} \cdot \frac{\partial \overline{x}^\beta}{\partial x^j} \cdot \frac{\partial x^k}{\partial \overline{x}^\nu}$$

Thus $C_j^{ij} = A_k^{ij} + B_k^{ij}$

is mixed tensor of order (2+1).

Subtraction : The difference of two tensors of the same rank and type is also a tensor of the same rank and type. Thus if A_k^{ij} and B_k^{ij} are tensors, then

$D_k^{ij} = A_k^{ij} - B_k^{ij}$ is also a tensor.

Outer product : The product of two tensors is a tensor whose rank is the sum of the ranks of the given tensors. This product which involves ordinary multiplication of the components of the tensor is called the outer product.

For example :

$$C_k^{ij} = A_k^i B^j$$

is the product of A_k^i and B^j. To prove this, let the transformation laws of A_k^i and B^j be

$$\overline{A}_\beta^\alpha = A_k^i \frac{\partial \overline{x}^\alpha}{\partial x^i} \frac{\partial x^k}{\partial \overline{x}^\beta}$$

$$\overline{B}^\nu = B^j \frac{\partial \overline{x}^\nu}{\partial x^j}$$

Then, on taking outer product, we find that

$$\overline{C}_\beta^{\alpha\nu} = \overline{A}_\beta^\alpha \overline{B}^\nu = A_k^i B^j \frac{\partial \overline{x}^\alpha}{\partial x^i} \cdot \frac{\partial x^k}{\partial \overline{x}^\beta} \cdot \frac{\partial \overline{x}^\nu}{\partial x^j}$$

$$\overline{C}_\beta^{\alpha\nu} = C_k^{ij} \frac{\partial \overline{x}^\alpha}{\partial x^i} \frac{\partial \overline{x}^\nu}{\partial x^j} \frac{\partial x^k}{\partial \overline{x}^\beta}$$

is a tensor of order 2 +1.

Contraction: If one contravariant and one convariant index of a tensor are set equal, the result indicates that a summation over the equal indices is to be taken according to the summation convention. This resulting sum is a tensor of rank two less than that of the original tensor. The process is called contraction.

The contraction of a tensor reduces the rank of the tensor by two, one in contraviant and one in covariant.

Inner Product : The inner product of two tensors is their outer product followed by contraction.

For example, if A^i and B_j are two first order tensors then $C_i^i = A^i B_i$ is their inner product.

The operations of addition, multiplication, etc. of relative tensors are the same as those of absolute tensors.

2.8 : Quotient rule

Consider a set of n^3 functions $A\ (ijk)$ with the indices i, j, k each ranging over 1, 2, . . ., n. Although the set of functions $A(i, j, k)$ has the right number of components, we know do not know whether it is tensor or not. Now, suppose that we something about the nature of the product of $A\ (i, j, k)$ with an arbitrary tensor. Then there is a theorem which enables us to establish whether $A\ (i, j, k)$ is a tensor without going to the trouble of determining the law of transformation directly.

Theorem 2.8.1 : If the inner product of a quantity X with an arbitrary tensor is a tensor then X is also a tensor.

We will prove the above theorem for the case of a tensor of order three.

Let $A\ (i,\ j,\ k)$ be a set of n^3 functions, B^k be an arbitrary contravariant tensor of order one. Let us suppose that the product $A\ (i, j, k)\ B^k$ is known to yield a tensor of the type C^{ij} i.e.

$$A\ (i, j, k)\ B^k = C^{ij}$$

then we can prove that $A\ (i, j, k)$ is a tensor of the type A_k^{ij}.

Since C^{ij} is a contravariant tensor of the second order therefore

$$\overline{C}^{pq} = C^{ij}\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j}$$

$$\Rightarrow \overline{C}^{pq} = A(i\ j\ k)B^k\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j}$$

$$\Rightarrow \overline{C}^{pq} = A(i\ j\ k)\overline{B}^r\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^r}$$

However

$$\overline{C}^{pq} = \overline{A}(pqr)\overline{B}^r$$

Comparing above two equations we get

$$\overline{A}(pqr)\overline{B}^r = A(i\,j\,k)\overline{B}^r\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^r}$$

$$\Rightarrow \left[\overline{A}(pqr) - A(i\,j\,k)\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^j}\frac{\partial x^k}{\partial \overline{x}^r}\right]\overline{B}^r = 0$$

Now $\overline{B}^r$ is arbitrary, hence the quantity within the bracket must vanish. Therefore,

$$\overline{A}(pqr) = A(i\,j\,k)\frac{\partial \overline{x}^p}{\partial x^i}\frac{\partial \overline{x}^q}{\partial x^i}\frac{\partial x^k}{\partial \overline{x}^r}$$

which is the law of transformation of the tensor of the type $\overline{A}_r^{pq}$. Hence $A\ (i\,j\,k)$ is a tensor of the type A_k^{ij}.

Examples :

Ex. 2.8.1: The components of a contravariant vector in x-coordinate system are 5 and 2. Obtain its components in the $\overline{x}$ - coordinate system if $\overline{x}^1 = 6(x^1)^2$ and $\overline{x}^2 = 4(x^1)^2 + 3(x^2)^2$.

Solution : By the transformation law of a contravariant vector, we have

$$\overline{A}^p = A^i\frac{\partial \overline{x}^p}{\partial x^i}$$

$$\therefore \overline{A}^1 = A^1 \frac{\partial \overline{x}^1}{\partial x^1} + A^2 \frac{\partial \overline{x}^1}{\partial x^2}$$

$$= 5(12x^1) + 2(0) = 60x$$

$$\overline{A}^2 = A^1 \frac{\partial \overline{x}^2}{\partial x^1} + A^2 \frac{\partial \overline{x}^2}{\partial x^2}$$

$$= 5\ (8x^1) + 2\ (6x^2)$$

$$= 40x^1 + 12\ x^2.$$

Ex. 2.8.2 : Show that the velocity of a fluid at any point is a contravariant tensor of rank one.

Solution : Let the coordinates of a point in the fluid be $x^i(t)$ at any time t. Then velocity v^i is given by

$$v^i = \frac{dx^i}{dt}$$

Let the coordinates be transformed to new coordinates $\overline{x}^\alpha$. In the transformed coordinates the velocity $\overline{v}^\alpha$ is given by

$$\overline{v}^\alpha = \frac{d\overline{x}^\alpha}{dt}.$$

$$= \frac{\partial \overline{x}^\alpha}{\partial x^i} \frac{dx^i}{dt}$$

$$\therefore = v^i \frac{\partial \overline{x}^\alpha}{\partial x^i}$$

which is the law of transformation of a contravariant tensor of rank one. Hence v^i is a contravariant tensor of rank one.

Ex. 2.8.3: Prove that Kronecker delta is a mixed tensor of rank two.

Solution : In the $\overline{x}$ - system by definition

$$\overline{\delta}^p_q = \frac{\partial \overline{x}^p}{\partial \overline{x}^q}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^i}{\partial \bar{x}^q} \qquad \text{(by chain rule)}$$

$$= \frac{\partial \bar{x}^p}{\partial x^i} \frac{\partial x^j}{\partial \bar{x}^q} \delta^i_j$$

$$\therefore \quad \bar{\delta}^p_q = \delta^i_j \frac{\partial \bar{x}^p}{\partial x^i} . \frac{\partial x^j}{\partial \bar{x}^q}$$

which is the law of transformation of a mixed tensor of rank two. Hence Kronecker delta is a mixed tensor of rank two.

Ex. 2.8.4: If $\bar{A}^i = A^\alpha \frac{\partial \bar{x}^i}{\partial x^\alpha}$, show that $A^i = \bar{A}^\alpha \frac{\partial x^i}{\partial \bar{x}^\alpha}$

Solution : Since $\bar{A}^i = A^\alpha \frac{\partial \bar{x}^i}{\partial x^\alpha}$

$$\Rightarrow \bar{A}^i \frac{\partial x^k}{\partial \bar{x}^i} = A^\alpha \frac{\partial \bar{x}^i}{\partial x^\alpha} \frac{\partial x^k}{\partial \bar{x}^i}$$

$$\Rightarrow \bar{A}^i \frac{\partial x^k}{\partial \bar{x}^i} = A^\alpha \frac{\partial x^k}{\partial x^\alpha}$$

$$\Rightarrow \bar{A}^i \frac{\partial x^k}{\partial \bar{x}^i} = A^k$$

$$\Rightarrow A^k = \bar{A}^i \frac{\partial x^k}{\partial x^i}$$

$$\Rightarrow A^i = \bar{A}^\alpha \frac{\partial x^i}{\partial \bar{x}^\alpha}$$

Ex.2.8.5: Show that every tensor can be expressed as the sum of two tensors, one of which is symmetric and the other skew-symmetric in a pair of covariant or contravariant indices.

Solution : Let A^{ij} be an arbitrary contravariant tensor of rank two. Now

$$A^{ij} = \frac{1}{2}\left(A^{ij} + A^{ji}\right) + \frac{1}{2}\left(A^{ij} - A^{ji}\right)$$

But $\quad B^{ij} = \frac{1}{2}\left(A^{ij} + A^{ji}\right) = B^{ji}$ is symmetric,

and $\quad C^{ij} = \frac{1}{2}\left(A^{ij} + A^{ji}\right) = -C^{ji}$ is skew-symmetric.

$\therefore A^{ij}$ can be expressed as the sum of a symmetric and a skew symmetric tensor.

Ex. 2.8.6: Prove that the equations of transformation of components of a contravariant vector possess the group property (or transitive property).

Solution : Let A^i and $\bar{A}^i$ be the components of a contravariant vector in the coordinate systems x^i and $\bar{x}^i$ respectively. If the coordinate system x^i are transformed to the coordinate system $\bar{x}^i$ then the component $\bar{A}^i$ are connected with components A^i by the relation

$$\bar{A}^i = A^j \frac{\partial \bar{x}^i}{\partial x^i} \qquad (2.8.1)$$

Multiplying (2.8.1) by $\frac{\partial x^k}{\partial \bar{x}^i}$, and summing for i, we get

$$\bar{A}^i \frac{\partial x^k}{\partial \bar{x}^i} = A^j \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial x^j} = A^j \delta^k_j = A^k$$

i.e. $$A^k = \bar{A}^i \frac{\partial x^k}{\partial \bar{x}^i} \qquad (2.8.2)$$

The relation (2.8.2) is the law of transformation for the components of a contravariant vector where the coordinate system $\bar{x}^i$ are transformed to the coordinate system x^i.

Comparing (2.8.1) and (2.8.2) we see that the relation between two sets of components is reciprocal one.

Let $\bar{\bar{A}}^i$ be the components of the same vector in the third co-ordinate system $\bar{\bar{x}}^i$. If the coordinates $\bar{x}^i$ are transformed to the coordinates $\bar{\bar{x}}^i$ then the law of transformation be

$$\bar{\bar{A}}^i = \bar{A}^j \frac{\partial \bar{\bar{x}}^i}{\partial \bar{x}^j} \qquad (2.8.9)$$

Substituting the value of $\overline{A}^j = A^k \frac{\partial \overline{x}^j}{\partial x^k}$ from (2.8.1) in (2.8.9) we get

$$\overline{A}^j = A^k \frac{\partial \overline{\overline{x}}^i}{\partial \overline{x}^j} \frac{\partial \overline{x}^j}{\partial x^k}$$

or, $$\overline{\overline{A}}^i = A^k \frac{\partial \overline{\overline{x}}^i}{\partial x^k}$$

This relation is of the same form as the law of transformation for the components of a covariant vector when the coordiante system x^i are transformed to the coordinate system $\overline{\overline{x}}^i$. Consequently the equations of transformation of components of a contravariant vector possess the group property (or transitive property).

2.9 Exercise for Unit - II :

2.1. If all the components of a tensor vanish in one coordinate system, then show that they necessarily vanish in all other admissible coordinate systems.

2.2. Prove that the equations of transformation of components of a covariant vector possess the group property (or transitive property).

2.3. If A^i_{jk} and B^i_{jk} are components of a tensor then prove that their sum and difference are also tensors.

2.4. Show that the outer product of two contravariant vectors is contravariant tensor of second order.

2.5. Prove that contraction of a mixed tensor yields a scalar invariant.

2.6. If A^{ij} are components of a contravariant tensor of order two and B_k are components of a covariant vector then prove that $A^{ij}B_k$ are components of a tensor of order three but $A^{ij}B_j$ are components of a tensor of order one.

2.7. If the sum A^iB_i is an invariant and if the quantities A^i are the components of an arbitrary contravariant vector then prove that quantities B_i are the components of a covariant vector.

2.8. State quotient law of tensors and use it to prove that Kronecker delta δ^i_j are components of mixed tensor of rank two.

2.9. If $\phi = a_{ij}A^iA^j$ then show that ϕ can always be written as $\phi = b_{ij}A^iA^j$ where b_{ij} is symmetric.

2.10. Show that the contraction of the outer product of the tensors A_i and B_j is an invariant.

2.11. A covariant tensor of order one has components $x^1 x^2$, $2x^3 - (x^2)^2$, $x^2 x^3$ in cartesian coordinates x^1, x^2, x^3. Find its covariant components in cylindrical coordinates.

2.12. The components of a contravariant tensor in the x - system are

$$A^{11} = 4, \qquad A^{12} = A^{21} = 0 \text{ and } A^{22} = 7.$$

Find its components in the $\bar{x}$- system, where

$$\bar{x}^1 = 4(x^1)^2 - 7(x^2)^2$$

$$\bar{x}^2 = 4x^1 - 5x^2 \cdot$$

2.10 Let Us Sum Up :

In this block, you have been introduced with scalars, vectors and different types of tensors. You have also learnt about the operations such as addition, subtraction, outer product, contration, inner product of various types of tensors. You have also learnt about quotient rule using which it becomes possible to check-up whether a given quantity is a vector or not.

BLOCK - 3

THE METRIC TENSOR

List of Contents :

3.0 Objective

After working with this block you will be able to

- find the base vectors in any coordinate system
- find the metric tensor in any curvilinear coordinate system
- find the physical components of a tensor

3.1 Introduction

In our discussion here we have included the difinition of metric tensors and derivation of the expressions for the length of a curve, magnititude of a vector, angle between two vectors and vector algebra is tensor notation Expressions for physical components of various order tensors will also be derived.

3.2 The Metric Tensor

Consider two infinitely closed points $A(x^1, x^2, x^3)$ and $A^1(x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$ in space. These points define an infinitesimal vector $d\vec{r}$, which is independent of the choice of co-ordinate ssytem. Let the length of the vector $d\vec{r}$ be denoted by ds. If $\vec{e}$ defines the unit vector directed along the straight line AA_1 then

$$d\vec{r} = ds\vec{e}. \tag{3.1}$$

From the point $A\,(x^1, x^2, x^3)$ we draw co-ordinate lines which do not lie in the same plane and are not, in general, orthogonal. Let $\vec{e}_n$ denote a system of covariant base vectors, not of unit length, directed along the tangents to the coordinate lines; then

$$d\vec{r}_1 = dx^1\vec{e}_1, \quad d\vec{r}_2 = dx^2\vec{e}_2, \quad d\vec{r}_3 = dx^3\vec{e}_3 \tag{3.2}$$

where $d\vec{r}_1, d\vec{r}_2, d\vec{r}_3$ are infinitesimal vectors defining a parallelopiped whose diagonal is the vector $d\vec{r}$, i.e.,

$$d\vec{r} = ds\vec{e} = dx^n\vec{e}_n = dx^1\vec{e}_1 + dx^2\vec{e}_2 + dx^3\vec{e}_3 \tag{3.3}$$

where $$\vec{e}_k = \frac{\partial \vec{r}}{\partial x^k} \tag{3.4}$$

From this, according to the rules for scalar multiplication of vectors, we find

$$ds^2 = \left(dx^n \vec{e}_n . dx^k \vec{e}_k \right) = g_{nk} dx^n dx^k \tag{3.5}$$

where

$$g_{nk} = \left(\vec{e}_n . \vec{e}_k \right), \quad (n, k = 1, 2, 3) \tag{3.6}$$

The coefficient g_{nk} in the quadratic form of the differentials dx^m as seen from (3.6) form of symmetric matrix ($g_{nk} = g_{kn}$). Thus, by the quotient rule, g_{nk} are the components of a covariant tensor, called the covariant metric tensor.

We have already defined (in Unit-I) the reciprocal base vectors e^k by the formulas

$$\vec{e}_n . \vec{e}^{\,k} = \delta_n^k \qquad (3.7)$$

We represent the vectors $\vec{e}_n$ as a linear combination of the vectors $\vec{e}^{\,k}$:

$$\vec{e}_n = C_{nk}\, \vec{e}^{\,k} \qquad (3.8)$$

$$\Rightarrow \vec{e}_n . \vec{e}_m = C_{nk}\, \vec{e}^{\,k} . \vec{e}_m$$

$$\Rightarrow g_{nm} = C_{nk}\, \delta^k_m$$

$$\Rightarrow g_{nm} = C_{nm}$$

Consequently (3.8) becomes

$$\vec{e}_n = g_{nm} \vec{e}^{\,m} \qquad (3.9)$$

Hence

$$\left.\begin{aligned} \vec{e}_1 &= g_{11}\vec{e}^{\,1} + g_{12}\vec{e}^{\,2} + g_{13}\vec{e}^{\,3} \\ \vec{e}_2 &= g_{21}\vec{e}^{\,1} + g_{22}\vec{e}^{\,2} + g_{23}\vec{e}^{\,3} \\ \vec{e}_3 &= g_{31}\vec{e}^{\,1} + g_{32}\vec{e}^{\,2} + g_{32}\vec{e}^{\,3} \end{aligned}\right\} \qquad (3.10)$$

From (3.10), by Crammer's rule, we find

$$\vec{e}^{\,k} = \frac{cofactor\ of\ g_{nk}}{g} \vec{e}_n = g^{nk} \vec{e}_n, (g \neq 0) \qquad (3.11)$$

where

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} \qquad (3.12)$$

Taking the scalar multiplication of (3.11) with $\vec{e}^{\,j}$,

we have

$$\vec{e}^{\,j}.\vec{e}^{\,k} = g^{nk}\, \vec{e}^{\,j}.\vec{e}^{\,n}$$

$$\Rightarrow \vec{e}^{\,j}.\vec{e}^{\,k} = g^{nk}\, \delta^{j}_{n}$$

$$\therefore\ \vec{e}^{\,j}.\vec{e}^{\,k} = g^{jk} \qquad \text{. . . (3.13)}$$

From (3.11), we have $g^{nk} = \dfrac{cofactor\ of\ g_{nk}}{g}$ . . . (3.14)

From (3.13) we see that

$$g^{jk} = \vec{e}^{\,j}.\vec{e}^{\,k} = \vec{e}^{\,k}.\vec{e}^{\,j} = g^{kj}$$

Hence, g^{jk} are symmetric. Substituting the value of $\vec{e}^{\,k}$ from (3.11) in (3.7)

we get

$$\vec{e}_{n}\,(g^{mk}\,\vec{e}_{m}) = \delta^{k}_{n}$$

$$\Rightarrow g_{nm}.g^{mk} = \delta^{k}_{n} \qquad (3.15)$$

From this we conclude that g^{jk} are the components of a contravariant tensor. The tensor g^{jk} is called the inverse metric tensor or Contravatiant metric tensor. The components g^{jk} of this tensor can be calculated by means of (3.14).

3.3 Associated Tensors

If A^k are components of a contravariant vector, then its inner product with g_{ij} *i.e.,* $g_{ij}A^j$ are components of a contravariant vector which is called associate to A^i by means of fundamental tensor. It is usually denoted by A^i. Thus

$$A_i = g_{ij}\, A^j \qquad (3.16)$$

Similarly if A_k are components of a covariant vector, then $g^{ij} A_j$ are components of a contravariant vector which is called associated to A_i and is denoted by A^i. Thus

$$A^i = g^{ij}\, A_j \qquad (3.17)$$

Hence, A_k and A^k may conveniently be considered as different, respectively covariant and contravariant components of the same vector $\vec{A}$.

The superscript of any tensor can be lowered by means of the metric tensor g_{ij}. This process of obtaining the associate tensor by composition with one of the fundamental tensor is called lowering the superscript or raising the subscript.

3.4 Length of a Curve

If ds is the length of an infinitesimal vector $d\vec{r}$ then by (3.5) we have

$$ds^2 = g_{ij}dx^i dx^j$$

i.e. $$ds = \sqrt{g_{ij}dx^i dx^j}$$

Consequently,

$$\frac{ds}{dt} = \sqrt{g_{ij}\frac{dx^i}{dt}.\frac{dx^j}{dt}} \qquad (3.18)$$

Let s be the length of the arc goining the points which corresponds to the values t_0 and t_1 of the parameter t then

$$s = \int_{t_0}^{t_1} \sqrt{g_{ij}\frac{dx^i}{dt}.\frac{dx^j}{dt}} \qquad (3.19)$$

3.5 Magnitude of a Vector

If $\vec{A}$ is a vector then it can be expressed as

$$\vec{A} = A^i \vec{e}_i \qquad (3.20)$$

Now, $$|\vec{A}|^2 = \vec{A}.\vec{A} = (A^i\vec{e}_i).(A^j\vec{e}_j)$$

$$= A^i A^j g_{ij}$$

$$\therefore \left|\vec{A}\right| = \sqrt{g_{ij}A^i A^j} \qquad (3.21)$$

The vector $\vec{A}$ can also be written as

$$\vec{A} = A_k \vec{e}^k \tag{3.22}$$

Consequently

$$\therefore \left|\vec{A}\right| = \sqrt{g^{ij} A_i A_j} \tag{3.23}$$

Using (3.20) and (3.22)

$$|\vec{A}|^2 = \vec{A}.\vec{A} = (A^i \vec{e}_i).(A_j \vec{e}^j)$$

$$= A^i A_j d_i^j = A^i A_i$$

$$\therefore \left|\vec{A}\right| = \sqrt{A^i A_i} \tag{3.24}$$

Hence

$$\left|\vec{A}\right| = \sqrt{g_{ij} A^i A^j} = \sqrt{g^{ij} A_i A_j} = \sqrt{A_i A^i}$$

3.6 Angle Between Two Vectors

We know that if θ be the angle between two vectors $\vec{A}$ and $\vec{B}$ then

$$\vec{A}.\vec{B} = |\vec{A}|.|\vec{B}| \, Cos\theta$$

$$\therefore Cos\theta = \frac{\vec{A}.\vec{B}}{\left|\vec{A}\right|\left|\vec{B}\right|}$$

$$= \frac{\left(A^i \vec{e}_i\right).\left(B^j \vec{e}_j\right)}{\left|\vec{A}\right|\left|\vec{B}\right|}$$

$$Cos\theta = \frac{g_{ij} A^i B^j}{\sqrt{g_{lm} A^l A^m}\sqrt{g_{rs} B^r B^s}} \tag{3.25}$$

3.7 Vector Algebra in Tensor Notation

If $\vec{A}$ and $\vec{B}$ be two vectors then their scalar product can also written as

$$\vec{A}.\vec{B} = g^{ij} A_i B_j = A^i B_i = A_j B^j \qquad (3.27)$$

From this we obtain a condition for orthogonality of two vectors $\vec{A}$ and $\vec{B}$:

$$g_{ij} A^i A^j = g^{ij} A_i A_j = A^i B_i = A_j B^j = 0 \qquad (3.28)$$

The vectors product of $\vec{A}$ and $\vec{B}$ is given by

$$\vec{A}\times\vec{B} = (A^i \vec{e}_i) \times (B^j \vec{e}_j) = A^i B^j (\vec{e}_i \times \vec{e}_j)$$

$$= A^i B^j (g\varepsilon_{ijk} \vec{e}_k)$$

$$= g\varepsilon_{ijk} A^i B^i \vec{e}_k \qquad (3.29)$$

where $g = [\vec{e}^1\ \vec{e}^2\ \vec{e}^3]$.

The vector product of $\vec{A}$ and $\vec{B}$ can also be written as

$$\vec{A}\times\vec{B} = G\varepsilon^{ijk} A_i B_i \vec{e}_k.$$

where $G = [\vec{e}^1\ \vec{e}^2\ \vec{e}^3]$.

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors then the scalar triple product of these vectors is given as

$$\vec{A}.\vec{B}\times\vec{C} = \vec{A}.(\vec{B}\times\vec{C})$$

$$\Rightarrow \left[\vec{A}\ \vec{B}\ \vec{C}\right] = \left(\vec{A}.\vec{e}_i\right).\left(B^j \vec{e}_j \times C^k \vec{e}_k\right)$$

$$= (A^i \vec{e}_i).g\ \varepsilon_{jkl} B^j C^k \vec{e}^l$$

$$= A^i g\ \varepsilon_{jkl} B^j C^k$$

$$= g\varepsilon_{jkl} A^i B^j C^k \qquad (3.31)$$

Scalar triple product (also known as box product) can also be written as

$$[\vec{A}\ \vec{B}\ \vec{C}] = (A_i \vec{e}^i).(B_j \vec{e}^j \times C_k \vec{e}^k)$$

$$= (A_i\vec{e}^{\,i}).(G\,\varepsilon^{jkl}\,B_j\,C_k\,\vec{e}_l)$$
$$= G\varepsilon^{jki}\,A_i\,B_j\,C_k$$
$$= G\varepsilon^{ijk}\,A_i\,B_j\,C_k \qquad (3.32)$$

Here we have used the notation

$$g = [\,\vec{e}_1\,\vec{e}_2\,\vec{e}_3] \text{ and } G = [\,\vec{e}^{\,1}\,\vec{e}^{\,2}\,\vec{e}^{\,3}]$$

i.e. $$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{23} & g_{33} \end{vmatrix}$$

and $$G = \begin{vmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{vmatrix}$$

3.8 Physical Components of a Tensor

The base vectors $\vec{e}_i$ and $\vec{e}^{\,i}$ are in general not unit vectors. In fact, their lengths are

$$\left|\vec{e}^{\,i}\right| = \sqrt{g_{ii}}\,,\ \left|\vec{e}^{\,i}\right| = \sqrt{g^{ii}}\ ,\ i \text{ not summed}$$

Since we know that

$$\vec{v} = v^i\vec{e}_i = v_i\vec{e}^{\,i}$$

therefore, $$\vec{v} = \sum_{i=1}^{3} v^i\sqrt{g_{ii}}\,\frac{\vec{e}_i}{\sqrt{g_{ii}}} = \sum_{i=1}^{3} v_i\sqrt{g^{ii}}\,\frac{\vec{e}^{\,i}}{\sqrt{g^{ii}}}$$

$$\Rightarrow \vec{v} = \sum_{i=1}^{3} v^i\sqrt{g_{ii}}\,\hat{e}_i = \sum_{i=1}^{3} v_i\sqrt{g^{ii}}\,\hat{e}_i \qquad (3.33)$$

Then since $\hat{e}_i = \dfrac{\vec{e}_i}{\sqrt{g_{ii}}}$ and $\hat{e}^i = \dfrac{\vec{e}^{\,i}}{\sqrt{g^{ii}}}$ are unit vectors, all components $v^i\sqrt{g_{ii}}$ and $v_i\sqrt{g^{ii}}$ (i not summed) will have the same physical dimensions. It is seen that $v^i\sqrt{g_{ii}}$ are the component of $\vec{v}$ resolved in the direction of unit vectors $\hat{e}_i$ which are tangent to the coordinate lines; and that $v_i\sqrt{g^{ii}}$ are the components of $\vec{v}$ resolved in the direction of unit vectors $\hat{e}^i$ which are perpendicular to the coordinate planes. The components

$$v^i\sqrt{g_{ii}} \text{ and } v_i\sqrt{g^{ii}}\,, \quad i \text{ not summed,}$$

are called the physical components of the vector $\vec{v}$. They do not transform according to the tensor transformation law and are not components of tensors.

The physical components $v^i\sqrt{g_{ii}}$ and $v_i\sqrt{g^{ii}}$, (i not summed), are denoted by $v^{(i)}$ and $v_{(ii)}$ respectively.

The physical components of second order tensor will be defined in terms of the physical components of a mixed tensor, and the physical components of a mixed tensor will be defined from a tensor equation involving a mixed tensor and contravariant vectors. For example, the inner product of a second order mixed tensor and contravariant vector is given by

$$\tau^i = T^i_j\, n^j \tag{3.34}$$

Substituting the physical components of τ^i and n^j, we get

$$\frac{\tau^{(i)}}{\sqrt{g_{ii}}} = \sum_{j=1}^{3} T^i_j \frac{n^{(j)}}{\sqrt{g_{jj}}}$$

or $$\tau^{(i)} = \sum_{j=1}^{3} T^i_j \frac{\sqrt{g_{ii}}}{\sqrt{g_{jj}}} n^{(j)} = \sum_{j=1}^{3} T^{(i)}_{(j)} n^{(j)}$$

where $$T^{(i)}_{(j)} = \frac{\sqrt{g_{ii}}}{\sqrt{g_{jj}}} T^i_j$$

is the physical component corresponding to T^i_j. Hence, the tensorial components of a mixed tensor are related to its physical components by

$$T^i_j = \frac{\sqrt{g_{jj}}}{\sqrt{g_{ii}}} T^{(i)}_{(j)} \tag{3.35}$$

where i, j are free indices and are not summed.

The physical components of other second order tensors are related to $T^{(i)}_{(j)}$ by relations

$$T^{ij} = g^{ik} T^j_k = \sum_{k=1}^{3} \sqrt{\frac{g_{kk}}{g_{jj}}}\, T^{(j)}_{(k)}\, g^{ik} \tag{3.36}$$

$$T_i^j = g_{ik} T^k_l g^{lj} = \sum_{k=1}^{3} \sum_{l=1}^{3} g_{ik} \sqrt{\frac{g_{ll}}{g_{kk}}}\, T^{(k)}_{(l)}\, g^{lj} \tag{3.37}$$

$$T_{ij} = g_{ik} T^k_j = \sum_{k=1}^{3} g_{ik} \sqrt{\frac{g_{jj}}{g_{kk}}}\, T^{(k)}_{(j)} \tag{3.38}$$

In orthogonal coordinates, a basis $\vec{e}_i$ and its reciprocal $\vec{e}^{\,i}$ are identical in directions and the unit base vectors are given by

$$\hat{e}_i = \frac{\vec{e}_i}{h_i} = h_i \vec{e}^{\,i}$$

where $h_i = \sqrt{g_{ii}}$, i not summed.

In this case $v^{(i)} = v_{(i)}$ and hence we denote the physical components of a vector $\vec{v}$ by $v(i)$.

Thus,

$$v^{(i)} = v^i \sqrt{g_{ii}}, \quad v_{(i)} = v_i \sqrt{g^{ii}}$$

$$\Rightarrow v^{(i)} = h_i v^i, \quad v(i) = v_i \frac{1}{h_i}$$

$$\Rightarrow v^{(i)} = \frac{v(i)}{h_i}, \quad v_i = h_i v(i) \tag{3.39}$$

where i is not to be summed.

The physical components of a second order tensor T are denoted in orthogonal coordinates by $T(ij)$.

The relations (3.35) to (3.38) reduce to

$$T^i_j = \frac{h_j}{h_i} T(ij) \tag{3.40}$$

$$T^i_j = \sum_{k=1}^{3} \sum_{l=1}^{3} g_{ik} \sqrt{\frac{g_{ll}}{g_{kk}}}\, T^{(k)}_{(l)} g^{lj}$$

$$= g_{ii} \sqrt{\frac{g_{jj}}{g_{ii}}}\, T^{(i)}_{(j)} g^{jj}$$

$$= h_i^2 \frac{h_j}{h_i} T^{(i)}_{(j)} \frac{1}{h_j^2}$$

$$T_i^j = \frac{h_i}{h_j} T(ij) \tag{3.41}$$

$$T^{ij} = \frac{1}{h_i h_j} T(ij) \tag{3.42}$$

$$T_{ij} = h_i h_j\, T(ij) \tag{3.43}$$

3.9 Examples:

3.1 : Find the metric of a Euclidean space referred to

(a) Cartesian Coordinates

(b) Cylindrical Coordinates

Solution:

(a) In cartesian coordinates we know that

$$\vec{e}_1 = \hat{i}, \quad \vec{e}_2 = \hat{j}, \quad \vec{e}_3 = \hat{k}$$

$$\therefore g_{11} = \vec{e}_1 . \vec{e}_1 = \hat{i} . \hat{i} = 1$$

$$g_{12} = \vec{e}_1 . \vec{e}_2 = \hat{i} . \hat{j} = 0.$$

Similarly $\quad g_{22} = g_{23} = 1,\ g_{13} = g_{31} = g_{21} = g_{23} = g_{32} = 0$

Therefore, the expression for the metric

$$ds^2 = g_{ij}\ dx^i\ dx^j$$

becomes

$$ds^2 = dx^2 + dy^2 + dz^2$$

This is the metric referred to cartesian coordinates.

(b) The cartesian coordinates x, y, z and cylindrical coordinates r, θ, z are related by

$$x = r\ cos\theta, \qquad y = r\ sin\theta, \qquad z = z$$

Let $\vec{R}$ denote the position vector of a point P.

Then

$$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{R} = r\ cos\theta\hat{i} + r\ sin\theta\ \hat{j} + z\hat{k}$$

$$\therefore \quad \vec{e}_r = \frac{\partial \vec{R}}{\partial r}$$

$$\Rightarrow \vec{e}_r = \cos\theta\ \hat{i} + \sin\theta\ \hat{j}$$

$$\vec{e}_\theta = \frac{\partial \vec{R}}{\partial \theta} = -r\ sin\theta\ \hat{i} + r\ cos\theta\ \hat{j}$$

$$\vec{e}_z = \frac{\partial \vec{R}}{\partial z} = \hat{k}$$

$$\therefore \quad g_{rr} = \vec{e}_r . \vec{e}_r = \cos^2\theta + \sin^2\theta = 1$$

$$g_{\theta\theta} = \vec{e}_\theta . \vec{e}_\theta = (-r\ \sin\theta)^2 + (r\ \cos\theta)^2 = r^2$$

$$g_{zz} = \vec{e}_z . \vec{e}_z = 1^2 = 1$$

$$g_{r\theta} = \vec{e}_r . \vec{e}_\theta = -r\ \sin\theta\ \ \cos\theta + r\ \cos\theta \sin\theta = 0$$

$$g_{\theta z} = \vec{e}_\theta . \vec{e}_z = 0, \quad g_{rz} = 0$$

Therefore, the expression for the metric

$$ds^2 = g_{ij}\ dx^i\ dx^j$$

becomes

$$ds^2 = g_{rr}\ dr^2 + g_{\theta\theta}\ d\theta^2 + g_{z2}\ dz^2 + 2g_{r\theta}\ drd_\theta + 2g_{\theta z}\ d\theta dz + 2g_{zr}\ dz\ dr$$

$$\Rightarrow ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

This is the metric referred to cylindrical coordinates.

3.2 : Calculate the fundamental tensor g_{ij} and the conjugate metric tensor for the given metric

$$ds^2 = 2(dx^1)^2 + 3(dx^2)^2 + 5(dx^3)^2 + 4dx^1dx^2 - 6dx^2\ dx^3 + 2dx^1\ dx^3.$$

Solution: The line element can be written as

$$\begin{aligned} ds^2 = 2(dx^1)^2 + 2dx^1dx^2 + dx^1dx^3 \\ + 2dx^2dx^1 + 3(dx^2)^2 - 3dx^2dx^3 \\ + dx^3dx^1 - 3dx^3dx^2 + 5(dx^3)^2 \end{aligned}$$

Thus metric tensor is given by

$$g_{ij} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 3 & -3 \\ 1 & -3 & 5 \end{bmatrix}$$

$$\therefore\ g = |g_{ij}| = 2(15 - 9) + 2\ (-3 - 10) + 1\ (-6 - 3)$$

$$= 12 - 26 - 9 = -23$$

The conjugate metric tensor g^{ij} is defined by

$$g^{ij} = \frac{\text{Cofactor of } g_{ij} \text{ in } |g_{ij}|}{\text{g}}$$

$$\therefore \quad g^{11} = \frac{6}{-23}, \quad g^{12} = \frac{-13}{-23}, \quad g^{13} = \frac{-9}{-23}$$

$$g^{21} = \frac{-13}{-23}, \quad g^{22} = \frac{9}{-23}, \quad g^{23} = \frac{8}{-23}$$

$$g^{31} = \frac{-9}{-23}, \quad g^{32} = \frac{8}{-23}, \quad g^{33} = \frac{2}{-23}$$

Therefore,

$$g^{ij} = \begin{bmatrix} \frac{-6}{23} & \frac{13}{23} & \frac{9}{23} \\ \frac{13}{23} & \frac{-9}{23} & \frac{-8}{23} \\ \frac{9}{23} & \frac{-8}{23} & \frac{-2}{23} \end{bmatrix}.$$

3.3 : Prove that $\sqrt{g}$ is a relative tensor of weight one.

Solution: We know that

$$g = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}$$

and also that the elements of the determinant g satisfy

$$\bar{g}_{pq} = g_{ij} \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q}$$

Taking determinant of both sides we get

$$\bar{g} = \left| \frac{\partial x^i}{\partial \bar{x}^p} \right| \left| \frac{\partial x^j}{\partial \bar{x}^q} \right| g$$

$$\Rightarrow \bar{g} = J^2 g$$

$$\Rightarrow \sqrt{\bar{g}} = J^1 \sqrt{g}$$

which shows that $\sqrt{g}$ is a relative tensor of weight one.

3.4 : Find the physical components of velocity of a particle in spherical coordinates.

Solution : We know that the velocity of a particle at any point is a contravariant tensor of order one.

The contravariant components of the velocity are

$$\frac{dx^1}{dt} = \frac{dr}{dt}, \frac{dx^2}{dt} = \frac{d\theta}{dt}, \frac{dx^3}{dt} = \frac{d\phi}{dt}$$

Therefore, the physical components of the velocity are

$$\sqrt{g_{11}}\frac{dx^1}{dt} = \sqrt{g_{rr}}\frac{dr}{dt} = \frac{dr}{dt}$$

$$\sqrt{g_{22}}\frac{dx^2}{dt} = \sqrt{g_{\theta\theta}}\frac{d\theta}{dt} = r\frac{d\theta}{dt}$$

$$\sqrt{g_{33}}\frac{dx^3}{dt} = \sqrt{g_{\phi\phi}}\frac{d\phi}{dt} = r\sin\theta\frac{d\phi}{dt}$$

(Here we have used the results $g_{rr} = 1$, $g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin^2\theta$)

3.5 : Prove that g is a relative tensor of weight two.

Solution : The elements of the determinant g given by g_{pq} transform according to

$$\bar{g}_{jk} = g_{pq}\frac{\partial x^p}{\partial \bar{x}^j}\frac{\partial x^q}{\partial \bar{x}^k}$$

Taking determinant of both sides, we get

$$\bar{g} = g\left|\frac{\partial x^p}{\partial \bar{x}^j}\right|\left|\frac{\partial x^q}{\partial \bar{x}^k}\right|$$

$$\bar{g} = g\,J^2$$

which shows that g is a relative tensor of weight two.

3.10 Exercise for Unit III

3.1. Prove that for an orthogonal coordinate system,

$$g_{12} = g_{23} = g_{31} = 0.$$

3.2. Prove that for an orthogonal coordinate system

$$g_{11} = \frac{1}{g^{11}},\ g_{22} = \frac{1}{g^{22}},\quad g_{33} = \frac{1}{g^{33}}$$

3.3. In a two dimensional space, find the quantities g^{ij}, if $g_{ij} = i + j$.

3.4. Find g_{ij}, g^{ij} and g corresponding to the metric

$$ds^2 = 5(dx^1)^2 + 3(dx^2)^2 + 4(dx^3)^2 - 6dx^1\, dx^2 + 4dx^2dx^3$$

3.5. If $A_j = g_{jk} A^k$ then prove that $A^k = g^{jk} A_j$.

3.6. Find the metric of an Euclidean space referred to spherical coordinates.

3.7. Corresponding to the metric $ds^2 = (dx^1)^2 + 2(dx^2)^2 + 3(dx^3)^2 - 8dx^2 dx^3$ evaluate g and g^{ij}.

3.8. Find the square of the element of arc in cyclindrical coordinates.

3.9. Prove that a cylindrical coordinate system is orthogonal.

3.10. Prove that

(i) $\det [g_{ij}] = J^2$

(ii) $\det [g^{ij}] = \dfrac{1}{J^2}$

3.11. Bipolar coordinates (θ,ϕ) in two dimensions are related to Cartesian coordinates (x,y) by

$$x = \frac{\sinh \theta}{\cosh \theta + \cos \phi},$$

$$y = \frac{\sin \theta}{\cosh \theta + \cos \phi}$$

(a) Find the covariant base vectors $\vec{e}_\theta$ and $\vec{e}_\theta$.

(b) Are the axes of the curvilinear coordinate system (θ, ϕ) orthogonal?

(c) Calculate the covariant metric tensor g_{ij}.

3.12. For the transformation:

$$\bar{x}^1 = 3x^1 + x^2 + 2x^3$$

$$\bar{x}^2 = x^1 + x^2 + 3x^3$$

$$\bar{x}^3 = 2x^1 - 3x^2 + x^3$$

find the following:

(i) The Jacobian of transformation and explicitly write out the inverse transformation.

(ii) The covariant base vectors $\vec{e}_j$.

(iii) Contravariant base vectors $\vec{e}^j$.

(iv) metric tensors g_{ij} and g^{ij},

(v) The expression for the line element.

Here (x^1, x^2, x^3) are Cartesian coordinates and $\left(\overline{x}^1, \overline{x}^2, \overline{x}^3\right)$ are curvilinear coordinates.

3.13. For the transformation of coordinates

$$x = y^1 + y^2 + y^3$$
$$y = y^1 - y^3$$
$$z = 2y^2 - 3y^3$$

find the

(i) covaraint base vectors

(ii) contravariant base vectors

(iii) metric tensors g_{ij} and g^{ij}.

3.14. Calculate the metric tensor g_{ij} for the curvilinear coordinates (y^1, y^2, y^3) if

$$x = y^1 y^2 y^3$$

$$y = y^1 y^2 \sqrt{1-\left(y^3\right)^2}$$

$$z = \frac{1}{2}\left[\left(y^1\right)^2 - \left(y^2\right)^2\right].$$

3.11 : Let Us Sum Up

In this block you have learnt how to find the find base vectors and metric tensor is various coordinate system. You have also learnt how to write the expressions for the length of a vector, angle between two vectors and the vector algebra in tensor notation. You have also learnt the method of finding the physical components of different types of tensors.

BLOCK - 4

CHRISTOFFEL SYMBOLS

List of Contents :

4.0 : Objective

After working with this block you will be able to

- differentiate the base vectors
- find Christoffel symbols of the first kind and second kind in any coordinate system.
- find Christoffel symbols in orthogonal coordinate system.

4.1 Introduction :

In our discussion here we have included the definitions of Christoffel symbols of the first kind and the second kind and the methods to derive them. Some examples are also included to derive to the expressions for Christoffel symbols in orthogonal coordinate systems.

4.2 Differentiation of Base Vectors

The base vectors of a curvilinear coordinates system are point functions which changes their directions from point to point. Since we know that any vector $\vec{A}$ can be written as

$$\vec{A} = A^j \vec{e}_j = A_j \vec{e}^j$$

therefore, if we form differential of $\vec{A}$, we get

$$d\vec{A} = \vec{e}_j \, d A^j + A^j d \vec{e}_j \tag{4.1}$$

$$d\vec{A} = \vec{e}^j \, d A_j + A_j d \vec{e}^j \tag{4.2}$$

In order to compute $d\vec{A}$ we must obtain formula for $d\vec{e}_j$ and $d\vec{e}^j$.

Since any vector can be expressed as a linear combination of base vectors, the derivative of base vectors $\dfrac{\partial \vec{e}_j}{\partial x^i}$ can be expressed as

$$\frac{\partial \vec{e}_j}{\partial x^i} = \Gamma^1_{ij} \vec{e}_1 + \Gamma^2_{ij} \vec{e}_2 + \Gamma^3_{ij} \vec{e}_3 \tag{4.3}$$

or,
$$\frac{\partial \vec{e}_j}{\partial x^i} = \Gamma^k_{ij} \vec{e}_k \tag{4.4}$$

The relation (4.4) gives

$$d\vec{e}_j = \frac{\partial \vec{e}_j}{\partial x^i} dx^i = \Gamma^k_{ij} dx^i \vec{e}_k \tag{4.5}$$

The quantities Γ^k_{ij} are called Christoffel symbols of the second kind. If the coordinates are Cartesian, then $\vec{e}_j$ are costant vectors, hence (4.4) gives $\Gamma^k_{ij} = 0$ while for a curvilinear coordinate system $\Gamma^k_{ij} \neq 0$.

Since we know that

$$\vec{e}_j = \frac{\partial \vec{r}}{\partial x^j} \tag{4.6}$$

therefore, $$\frac{d\vec{e}_j}{dx^i} = \frac{\partial}{\partial x^i}\left(\frac{\partial \vec{r}}{\partial x^j}\right) = \frac{\partial^2 \vec{r}}{\partial x^i \partial x^j}$$

$$= \frac{\partial^2 \vec{r}}{\partial x^j \partial x^i} = \frac{\partial}{\partial x^j}\left(\frac{\partial \vec{r}}{\partial x^i}\right)$$

$$\Rightarrow \frac{\partial \vec{e}_j}{\partial x^i} = \frac{\partial \vec{e}_i}{\partial x^j}$$

$$\Rightarrow \Gamma^k_{ij}\vec{e}_k = \Gamma^k_{ji}\vec{e}_k \qquad \text{(using 4.4)}$$

$$\Rightarrow \Gamma^k_{ij} = \Gamma^k_{ji} \qquad (4.7)$$

In some books Γ^k_{ij} is also denoted as $\left\{ {k \atop ij} \right\}$. From (4.4) we have

$$\frac{\partial \vec{e}_j}{\partial x^i} = \Gamma^l_{ij}\vec{e}_l$$

therefore, $$\vec{e}^{\,k} \cdot \frac{\partial \vec{e}_j}{\partial x^i} = \Gamma^l_{ij}\vec{e}^{\,k} \cdot \vec{e}_l$$

$$\Rightarrow \vec{e}^{\,k} \cdot \frac{\partial \vec{e}_j}{\partial x^i} = \Gamma^l_{ij}\delta^k_l$$

$$\Rightarrow \vec{e}^{\,k} \cdot \frac{\partial \vec{e}_j}{\partial x^i} = \Gamma^k_{ij}$$

$$\therefore \Gamma^k_{ij} = \vec{e}^{\,k} \cdot \frac{\partial \vec{e}_j}{dx^i} \qquad (4.8)$$

This expresses the Christoffel symbols in terms of base vectors.

To prove:

$$\frac{\partial \vec{e}^{\,i}}{\partial x^k} = -\vec{e}^{\,j}\,\Gamma^i_{jk}$$

Proof:

We know that

$$\vec{e}_j . \vec{e}^{\,i} = \delta^i_j$$

$$\Rightarrow \frac{\partial \vec{e}_j}{\partial x^k} . \vec{e}^{\,i} + \vec{e}_j . \frac{\partial \vec{e}^{\,i}}{\partial x^k} = 0$$

$$\Rightarrow \vec{e}^{\,i} . \frac{\partial \vec{e}_j}{\partial x^k} = -\vec{e}_j . \frac{\partial \vec{e}^{\,i}}{\partial x^k}$$

$$\Rightarrow \Gamma^i_{jk} = -\vec{e}_j . \frac{\partial \vec{e}^{\,i}}{\partial x^k} \qquad \text{(Using 4.8)}$$

$$\Rightarrow \frac{\partial \vec{e}^{\,i}}{\partial x^k} . \vec{e}_j = -\Gamma^i_{jk} \qquad (4.9)$$

We now represent $\frac{\partial \vec{e}^{\,i}}{\partial x^k}$ as

$$\frac{\partial \vec{e}^{\,i}}{\partial x^k} = B^i_{mk} \vec{e}^{\,m} \qquad (4.10)$$

Multiplying both sides of this equality scalarly by $\vec{e}_j$ we get

$$\frac{\partial \vec{e}^{\,i}}{\partial x^k} \vec{e}_j = B^i_{mk} \vec{e}^{\,m} . \vec{e}_j$$

$$\Rightarrow -\Gamma^i_{jk} = B^i_{jk} \qquad \text{(using 4.9)}$$

Hence (4.10) takes the form

$$\frac{\partial \vec{e^i}}{dx^k} = -\Gamma^i_{jk} \vec{e^j} \qquad (4.11)$$

If we know the laws of transformation of $\vec{e}^k$ and $\vec{e}_j$ then the laws of transformation for

$$\Gamma^{i}_{jk} = \vec{e}^{\,k} \cdot \frac{\partial \vec{e}_j}{\partial x^i}$$

$$= \left(\frac{\partial x^k}{\partial \bar{x}^{\alpha}} \vec{e}^{\,\alpha}\right) \cdot \left[\frac{\partial}{\partial x^i}\left(\frac{\partial \bar{x}^{\beta}}{\partial x^j} \vec{e}_{\beta}\right)\right]$$

$$= \frac{\partial x^k}{\partial \bar{x}^{\alpha}} \frac{\partial \bar{x}\beta}{\partial x^j} \vec{e}^{\,\alpha} \cdot \frac{\partial \vec{e}_{\beta}}{\partial x^i} + \frac{\partial x^k}{\partial \bar{x}^{\alpha}} \frac{\partial^2 \bar{x}^{\beta}}{\partial x^i \partial x^j} \vec{e}^{\,\alpha} \cdot \vec{e}_{\beta}$$

$$= \frac{\partial x^k}{\partial \bar{x}^{\alpha}} \frac{\partial \bar{x}\beta}{\partial x^j} \vec{e}^{\,\alpha} \cdot \frac{\partial \vec{e}_{\beta}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial \bar{x}^{i}} + \frac{\partial x^k}{\partial \bar{x}^{\alpha}} \frac{\partial^2 \bar{x}^{\beta}}{\partial x^i \partial x^j} \delta^{\alpha}_{\beta}$$

$$\Gamma^{k}_{ij} = \frac{\partial x^k}{\partial \bar{x}^{\alpha}} \frac{\partial \bar{x}^{\beta}}{\partial x^j} \frac{\partial \bar{x}^{\nu}}{\partial x^i} \bar{\Gamma}^{\alpha}_{\beta\nu} + \frac{\partial x^k}{\partial \bar{x}^{\alpha}} \frac{\partial^2 \bar{x}^{\alpha}}{\partial x^i \partial x^j} \qquad (4.12)$$

The equation (4.12) shows that Γ^{k}_{ij} are not the components of a tensor except when

$$\frac{\partial x^k}{\partial \bar{x}^{\alpha}} \cdot \frac{\partial^2 \bar{x}^{\alpha}}{\partial x^i \partial x^j} = 0 \qquad (4.13)$$

4.3 Christoffel symbols in terms of metric tensors

We know that

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j$$

Defferentiating partially with respect to *k*, we get

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \vec{e}_i}{\partial x^k} \cdot \vec{e}_j + \vec{e}_l \cdot \frac{\partial \vec{e}_j}{\partial x^k}$$

$$\Rightarrow \frac{\partial g_{ij}}{\partial x^k} = \Gamma^{m}_{ik} \vec{e}_m \cdot \vec{e}_j + \vec{e}_i \cdot \Gamma^{n}_{jk} \vec{e}_n$$

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma^{m}_{ik}\, g_{mj} + \Gamma^{n}_{jk}\, g_{in} \qquad (4.14)$$

We denote $\Gamma^m_{ik}\, g_{mj}$ by $[ik, j]$ or $\Gamma ik, j$ and define it as Christoffel symbol of the first kind.

Thus (4.14) becomes

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ik,j} + \Gamma_{jk,i} \tag{4.15}$$

From this we get

$$\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ij,k} + \Gamma_{kj,i}$$

$$+ \Gamma_{ji,k} + \Gamma_{ki,j} - \Gamma_{ik,j} - \Gamma_{jk,i}$$

$$= 2\,\Gamma_{ij,k}$$

$$\therefore \Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{4.16}$$

By the definition of Christoffel symbols of the first kind we know that

$$g_{mk}\Gamma^m_{ij} = \Gamma_{ij,k} \tag{4.17}$$

Multiplying (4.17) by g^{nk} we get

$$g^{nk} g_{mk}\, \Gamma^m_{ij} = g^{nk}\Gamma_{ij,k}$$

$$\delta^n_m\, \Gamma^m_{ij} = g^{nk}\Gamma_{ij,k}$$

$$\Gamma^n_{ij} = g^{nk}\,\Gamma_{ij,k} \tag{4.18}$$

Thus, $\Gamma^n_{ij} = \frac{1}{2} g^{nk}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right).$ (4.19)

To prove $\frac{\partial}{\partial x^k}\left(\log \sqrt{g}\right) = \Gamma^l_{kl}$

Proof: We know that

$g = det\,(g_{ij})$ and $g^{ij} = \frac{Cofactor\ of\ g_{ij}}{g}$

i.e, Cofactor of $g_{ij} = g\, g^{ij}$

Therefore,

$$\frac{\partial g}{\partial x^k} = \frac{\partial g_{ij}}{\partial x^k} \quad gg^{ij}$$

$$= gg^{ij}\left(\Gamma_{ik,j} + \Gamma_{jk,i}\right)$$

$$= g\left(g^{ij}\Gamma_{ik,j} + g^{ij}\,\Gamma_{jk,i}\right)$$

$$= g\left(g^{ij}\Gamma^{m}_{ik}\,g_{mj} + g^{ij}\,\Gamma^{n}_{jk}\,g_{ni}\right)$$

$$\Rightarrow \frac{1}{g}\frac{\partial g}{\partial x^k} = \delta^{i}_{m}\Gamma^{m}_{ik} + \delta^{j}_{n}\Gamma^{n}_{jk}$$

$$= \Gamma^{i}_{ik} + \Gamma^{j}_{jk}$$

$$= \Gamma^{l}_{lk} + \Gamma^{l}_{lk}$$

$$\frac{1}{g}\frac{\partial g}{\partial x^k} = 2\Gamma^{l}_{kl}$$

$$\Rightarrow \Gamma^{l}_{kl} = \frac{1}{2g}\frac{\partial g}{\partial x^k}$$

$$\therefore\ \Gamma^{l}_{kl} = \frac{\partial}{\partial x^k}\left(\log\left(\sqrt{g}\right)\right) \tag{4.20}$$

To prove,

$$\frac{\partial g^{ik}}{\partial x^i} = -g^{hk}\,\Gamma^{i}_{hj} - g^{hi}\,\Gamma^{k}_{hj}$$

Proof: Since we know that

$$g^{ik} = \vec{e}^{\,i}.\,\vec{e}^{\,k}$$

Therefore,

$$\frac{\partial g^{ik}}{\partial x^j} = \frac{\partial \vec{e}^{\,i}}{\partial x^j}.\vec{e}^{\,k} + \vec{e}^{\,i}.\frac{\partial \vec{e}^{\,k}}{\partial x^j}$$

$$\Rightarrow \frac{\partial g^{ik}}{\partial x^j} = -\Gamma^{i}_{hj}\,\vec{e}^{\,h}.\vec{e}^{\,k} + \vec{e}^{\,i}\left(-\Gamma^{k}_{hj}\,\vec{e}^{\,h}\right) \qquad \text{(by using 4.11)}$$

$$\Rightarrow \frac{\partial g^{ik}}{\partial x^j} = -g^{hk}\Gamma^{i}_{hj} - g^{hi}\Gamma^{k}_{hj} \tag{4.21}$$

4.4 Christoffel symbols in orthogonal coordinate systems

In an orthogonal system of coordinates $g_{ij} = 0$ if $i \neq j$ and $g_{ii} = h_i^2$. When all the indices i, j and k are different, the formula

$$\Gamma_{ij,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \quad (4.16)$$

gives

$$\Gamma_{ij,k} = 0 \text{ when i, j, k are different.} \quad (4.22)$$

When $i = k$ and j is different, we get

$$\Gamma_{ij,i} = \frac{1}{2}\left(\frac{\partial g_{ii}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^i}\right)$$

$$= \frac{1}{2}\frac{\partial h_i^2}{\partial x^j}$$

$$\therefore \Gamma_{ij,i} = h_i \frac{\partial h_i}{\partial x^j} \quad (4.23)$$

In case $i = j = k$, the formula (4.23) works. It also works when $j = k$. The case when $i = j$ we have

$$\therefore \Gamma_{ii,k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^i} - \frac{\partial g_{ii}}{\partial x^k}\right)$$

$$= -\frac{1}{2}\frac{\partial h_i^2}{\partial x^k}$$

$(\because g_{ik} = g_{ki} = 0)$

$$\Gamma_{ii,k} = -h_i \frac{\partial h_i}{\partial x^k} \quad (4.24)$$

The christoffel symbols of second kind Γ_{ij}^k are given by the formula

$$\Gamma_{ij}^k = g^{kn}\Gamma_{ij,n} \quad (4.18)$$

In orthognal coordinates $g^{ii} = \frac{1}{h_i^2}$. Hence we get

$$\Gamma_{ij}^k = 0 \text{ when } i, j, k \text{ are all different.} \quad (4.25)$$

This case when $i = k$ and j is different, we have

$$\Gamma_{ij}^i = g^{in}\Gamma_{ij,n} = g^{ii}\Gamma_{ij,i}$$

$$= \frac{1}{h_i^2} h_i \frac{\partial h_i}{\partial x^j}$$

$$\therefore \Gamma^i_{ij} = \frac{1}{h_i} \frac{\partial h_i}{\partial x^j} \qquad (4.26)$$

This formula also holds good when $i = j$, and $j = k$ when i is different. The case when $i = j$ and k is different, we have

$$\Gamma^k_{ii} = g^{kn} \Gamma_{ii,n} = g^{kk} \Gamma_{ii,k}$$

$$= \frac{1}{h_k^2} \left(- h_i \frac{\partial h_i}{\partial x^k} \right)$$

$$\therefore \Gamma^k_{ii} = - \frac{h_i}{h_k^2} \frac{\partial h_i}{\partial x^k} \qquad (4.27)$$

4.5 Examples:

4.1. Determine the Christoffel symbols for the metric

$$ds^2 = \left(dx^1\right)^2 + \left[\left(x^2\right)^2 - \left(x^1\right)^2\right]\left(dx^2\right)^2$$

Solution:

Comparing the given metric with

$$ds^2 = g_{ij} dx^i dx^j$$

we get

$$g_{11} = 1, \quad g_{22} = \left(x^2\right)^2 - \left(x^1\right)^2, \quad g_{12} = g_{21} = 0$$

Here $g_{ij} = 0$ for $i \neq j$ therefore the coordinate system (x^1, x^2) is orthogonal. Now for orthogonal coordinate system the non-vanishing Christoffel symbols of the first kind are

$$\Gamma_{22,1} = -h_2 \frac{\partial h_2}{\partial x^1} = -\sqrt{g_{22}} \frac{\partial \sqrt{g_{22}}}{\partial x^1}$$

$$= \sqrt{\left(x^2\right)^2 - \left(x^1\right)^2} . \frac{1}{2\sqrt{\left(x^2\right)^2 - \left(x^1\right)^2}} . \left(-2x^1\right) = -x^1$$

$$\Gamma_{22,2} = h_2 \frac{\partial h_2}{\partial x^2} = \sqrt{\left(x^2\right)^2 - \left(x^1\right)^2} . \frac{1}{2\sqrt{\left(x^2\right)^2 - \left(x^1\right)^2}} . \left(2x^1\right) = x^1$$

$$\Gamma_{12,2}=\Gamma_{21,2}=h_2\frac{\partial h_2}{\partial x^1}=\sqrt{\left(x^2\right)^2-\left(x^1\right)^2}\cdot\frac{1}{2\sqrt{\left(x^2\right)^2-\left(x^1\right)^2}}\cdot\left(-2x^1\right)=-x^1$$

and the non-vanishing Christoffel symbols of the second kind are

$$\Gamma^1_{22}=-\frac{h_2}{h_1^2}\cdot\frac{\partial h_2}{\partial x^1}=-\left[\left(x^2\right)^2-\left(x^1\right)^2\right]^{\frac{1}{2}}\cdot\frac{1}{2\left[\left(x_2\right)^2-\left(x^1\right)^2\right]^{\frac{1}{2}}}\left(-2x^1\right)$$

$$\Rightarrow\Gamma^1_{22}=x^1$$

$$\Gamma^2_{22}=\frac{1}{h_2}\frac{\partial h_2}{\partial x^2}=\frac{1}{\sqrt{\left(x^2\right)^2-\left(x^1\right)^2}}\cdot\frac{1}{2\sqrt{\left(x^2\right)^2-\left(x^1\right)^2}}\cdot\left(2x^2\right)$$

$$\Rightarrow\Gamma^2_{22}=\frac{x^2}{\left(x^2\right)^2-\left(x^1\right)^2},$$

$$\Gamma^2_{12}=\Gamma^2_{21}=\frac{1}{h_2}\frac{\partial h_2}{\partial x^1}$$

$$=\frac{1}{\sqrt{\left(x^2\right)^2-\left(x^1\right)^2}}\cdot\frac{1}{2\sqrt{\left(x^2\right)^2-\left(x^1\right)^2}}\cdot\left(-2x^1\right)$$

$$=\frac{x^1}{\left(x^1\right)^2-\left(x^2\right)^2}$$

4.2. Determine the Christoffel symbols in the curvilinear coordinate system (u, v, z) where

$$x=\frac{1}{2}\left(u^2-v^2\right),\quad y=uv,\quad z=z$$

Solution: If $\hat{i},\hat{j},\hat{k}$ are unit vectors along x-axis, y-axis, z-axis in Carterian coordinate system (x,y,z) and $\vec{r}$ is the position vector of a point of P then

$$\vec{r}=x\hat{l}+y\hat{j}+z\hat{k}$$

$$\Rightarrow\vec{r}=\frac{1}{2}\left(u^2-v^2\right)\hat{l}+uv\hat{j}+z\hat{k}$$

If $\vec{e}_1,\vec{e}_2,\vec{e}_3$ are base vectors along u axis, v axis, z axis then we know that

$$\vec{e}_1,=\frac{\partial\vec{r}}{\partial u},\quad\vec{e}_2=\frac{\partial\vec{r}}{\partial v},\quad\vec{e}_3=\frac{\partial\vec{r}}{\partial z}$$

Therefore,

$$\vec{e}_1 = u\,\hat{i} + v\,\hat{J}$$
$$\vec{e}_2 = -v\,\hat{l} + u\,\hat{J}$$
$$\vec{e}_3 = \hat{k}$$

Since we know that $g_{ij} = \vec{e}_i.\ \vec{e}_j$ therefore,

$$g_{11} = \vec{e}_1.\vec{e}_1 = u^2 + v^2, \quad g_{22} = \vec{e}_2.\vec{e}_2 = u^2 + v^2,$$
$$g_{33} = \vec{e}_3.\vec{e}_3 = 1, \quad g_{12} = \vec{e}_1.\vec{e}_2 = u(-v) + v(u) = 0$$
$$g_{23} = \vec{e}_2.\vec{e}_3 = 0, \quad g_{31} = \vec{e}_3.\vec{e}_1 = 0$$

Hence metric tensor g_{ij} become

$$\left[g_{ij}\right] = \begin{bmatrix} u^2+v^2 & 0 & 0 \\ 0 & u^2+v^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Here we observe that $g_{ij} = 0$ when $i \neq j$ therefore the coordinate system (u, v, z) is orthogonal and hence

$$h_1 = \sqrt{g_{11}} = \sqrt{u^2+v^2}, \quad h_2 = \sqrt{g_{22}} = \sqrt{u^2+v^2},$$
$$h_3 = \sqrt{g_{33}} = 1,$$
$$g^{ii} = \frac{1}{g_{ii}} = \frac{1}{h_i^2},$$

$$\left[g^{ij}\right] = \begin{bmatrix} \frac{1}{u^2+v^2} & 0 & 0 \\ 0 & \frac{1}{u^2+v^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now the non-vanishing Christoffel symbols of the first kind are given by

$$\Gamma_{11,1} = h_1\frac{\partial h_1}{\partial x^1} = h_1\frac{\partial h_1}{\partial u} = \sqrt{u^2+v^2}.\frac{1}{2\sqrt{u^2+v^2}}.2u = u$$
$$\Gamma_{22,2} = h_2\frac{\partial h_2}{\partial x^2} = h_2\frac{\partial h_2}{\partial v} = \sqrt{u^2+v^2}.\frac{1}{2\sqrt{u^2+v^2}}.2v = v$$

$$\Gamma_{12,1} = \Gamma_{21,1} = h_1\frac{\partial h_1}{\partial x^2} = \sqrt{u^2+v^2}.\frac{1}{2\sqrt{u^2+v^2}}.2v = v$$

$$\Gamma_{21,2} = \Gamma_{12,2} = h_2 \frac{\partial h_2}{\partial x^1} = \sqrt{u^2+v^2}.\frac{1}{2\sqrt{u^2+v^2}}.2u = u$$

$$\Gamma_{11,2} = -h_1 \frac{\partial h_1}{\partial x^2} = -\sqrt{u^2+v^2}.\frac{1}{2\sqrt{u^2+v^2}}.2v = -v$$

$$\Gamma_{22,1} = -h_2 \frac{\partial h_2}{\partial x^1} = -\sqrt{u^2+v^2}.\frac{1}{2\sqrt{u^2+v^2}}.2u = -u$$

and the non vanishing Christoffel symbols of the second kind are given by

$$\Gamma^1_{11} = \frac{1}{h_1}\frac{\partial h_1}{\partial x^1} = \frac{1}{\sqrt{u^2+v^2}}.\frac{1}{2\sqrt{u^2+v^2}}.2u = \frac{u}{u^2+v^2}$$

$$\Gamma^2_{22} = \frac{1}{h_2}\frac{\partial h_2}{\partial x^2} = \frac{1}{\sqrt{u^2+v^2}}.\frac{1}{2\sqrt{u^2+v^2}}.2v = \frac{v}{u^2+v^2}$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{h_1}\frac{\partial h_1}{\partial x^2} = \frac{1}{\sqrt{u^2+v^2}}.\frac{1}{2\sqrt{u^2+v^2}}.2v = \frac{v}{u^2+v^2}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{h_2}\frac{\partial h_2}{\partial x^1} = \frac{1}{\sqrt{u^2+v^2}}.\frac{1}{2\sqrt{u^2+v^2}}.2v = \frac{u}{u^2+v^2}$$

$$\Gamma^1_{22} = -\frac{h_2}{h_2^1}\frac{\partial h_2}{\partial x^1} = \frac{\sqrt{u^2+v^2}}{u^2+v^2}.\frac{1}{2\sqrt{u^2+v^2}}.2u = \frac{-u}{u^2+v^2}$$

$$\Gamma^2_{11} = -\frac{h_1}{h_2^2}\frac{\partial h_1}{\partial x^2} = \frac{\sqrt{u^2+v^2}}{u^2+v^2}.\frac{1}{2\sqrt{u^2+v^2}}.2u = \frac{-v}{u^2+v^2}$$

In cartesian coordiante system $h_1 = h_2 = h_3 = 1$ which gives that Christoffel symbols vanish. In cylendrical polar coordiantes $h_1 = 1, h_2 = r$, $h_3 = 1$, hence the only non-vanishing christoffel symbols are

$$\Gamma_{r\theta,\theta} = \Gamma_{\theta r,\theta} = r$$

$$\Gamma_{\theta\theta,r} = \Gamma^r_{\theta\theta} = -r$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

In spherical polar coordinates $h_1 = 1$, $h_2 = r$, $h_3 = r \sin\theta$ and non-vanishing christoffel symbols are

$$\Gamma_{\theta r,\theta} = \Gamma_{r\theta,\theta} = -\Gamma_{\theta\theta,r} = r$$

$$\Gamma_{\phi r,\phi} = \Gamma_{r\phi,\phi} = -\Gamma_{\phi\phi,r} = r \sin^2\theta$$

$$\Gamma_{\phi\theta,\phi} = \Gamma_{\theta\phi,\phi} = -\Gamma_{\phi\phi,\theta} = r^2 \sin\theta \cos\theta$$

$$\Gamma^r_{\theta\theta} = -r, \quad \Gamma^r_{\phi\phi} = -r \sin^2\theta$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}$$

$$\Gamma^\theta_{\phi\phi} = \sin\theta \cos\theta, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta$$

4.6 Exercise:

4.1. Prove that christoffel sumbols of second kind are not a tensor, but if the laws of transformation of coordinates are linear they are components of a tensor of type (1,2).

4.2. Prove that both the christoffel symbols and $\Gamma_{ij,k}$ and Γ^k_{ij} are symmetric in i and j.

4.3. Prove that $\Gamma_{ij,k} = g_{kl}\Gamma^l_{ij}$

4.4. Determine the christoffel symbols of the first kind and the second kind in (a) rectangular, (b) cylindrical, and (c) spherical polar coordinates.

4.5. Determine the christoffel symbols of the first kind and second kind for the coordinate system (u, v, z) where $x = a$ coshu cos V, $y = a$ sinhu sin V, $z = z$.

4.6. Determine the christoffel symbols of the first kind and second kind for the coordinate system (u, v, ϕ) where $x = uv \cos\phi$, $y = uv \sin\phi$, $z = \frac{1}{2}\left(u^2 - v^2\right)$.

4.7. Calcualte the six christoffel symbols of the two dimensional uv-system defined by

$$x = 2\, e^{u-v}, \qquad y = -e^{3u+2v}.$$

4.7 Let Us Sum Up

In this block you have learnt how to compute Christiffel symbols of the first kind and the second kind in orthogonal coordinate systems. You have also learnt that Christoffel symbols, in general, are not tensor quantities.

BLOCK - 5

COVARIANT DIFFERENTIATION

List of Contents :

5.0 Objective

After working with this block you will be able to

- find covariant derivative of various types of tensor
- write the expressions of gradient, divergence, curl and Laplacian in any coordinate system.
- define the Riemann-Christoffel tensor

5.1 Introduction

In our discussion here we have included : the covariant derivative of tensors, rules of derivatives, gradient, divergence, curl and Laplacian in tensor notation. Certain examples related to them a will also be discussed.

The covariant derivative of tensors

Covariant derivative of a vector : Let (x^i) denote a system of curvilinear coordinates in the Euclidean 3-D space, and let $\vec{e}_i$ be the natural basis for the system. Consider a vector field defined by $\vec{A}$. At a point P, we have

$$\vec{A} = A^i \vec{e}_i \tag{5.1}$$

The differential $d\vec{A}$ is given by

$$d\vec{A} = d\left(A^i \vec{e}_i\right)$$

$$= \vec{e}_i \, dA^i + A^i d\,\vec{e}_i$$

$$= \vec{e}_i \, dA^i + A^j d\,\vec{e}_j$$

$$= \vec{e}_i \frac{\partial A^i}{\partial x^k} dx^k + A^j \frac{\partial \vec{e}_j}{\partial x^k} dx^k$$

$$= \vec{e}_i \frac{\partial A^i}{\partial x^k} dx^k + A^j \Gamma^i_{jk} \vec{e}_i \, dx^k$$

$$d\vec{A} = \left[\left(\frac{\partial A^i}{\partial x^k} + A^j \Gamma^i_{jk}\right) dx^k\right] \vec{e}_i \tag{5.2}$$

Let $$d\vec{A} = \delta A^i \vec{e}_i \tag{5.3}$$

then

$$\delta A^i = \left(\frac{\partial A^i}{\partial x^k} + A^j \Gamma^i_{jk}\right) dx^k$$

or, $$\delta A^i = A^i{}_{,k} \, dx^k \tag{5.4}$$

where, $$A^i{}_{,k} = \frac{\partial A^i}{\partial x^k} + A^j \Gamma^i_{jk} \tag{5.5}$$

Since δA^i are contravariant components of a vector, and dx^k are the contravariant components of any arbitrary vector hence we conclude from (5.4), by quotient rule, that $A^i{}_{,k}$ are the components of a second order mixed tensor. We define $A^i{}_{,k}$ as the covariant derivative of A^i.

Let $\vec{A} = A_i \vec{e}^i$ (5.6)

The differential $d\vec{A}$ is given by

$$d\vec{A} = \vec{e}^i \, dA_i + A_i \, d\vec{e}^i$$

$$= \vec{e}^i \, dA_i + A_j d\vec{e}^j$$

$$= \vec{e}^i \frac{\partial A_i}{\partial x^k} dx^k + A_j \frac{\partial \vec{e}^j}{\partial x^k} dx^k$$

$$= \vec{e}^i \frac{\partial A_i}{\partial x^k} dx^k + A_j \left(-\vec{e}^i \Gamma^j_{ik} \right) dx^k$$

$$d\vec{A} = \left[\left(\frac{\partial A_i}{\partial x^k} - A_j \Gamma^j_{ik} \right) dx^k \right] \vec{e}^i \tag{5.7}$$

Let $d\vec{A} = \delta \, A_i \vec{e}^i$ (5.8)

$$\therefore \delta \, A_i = \left(\frac{\partial A_i}{\partial x^k} - A_j \Gamma^j_{ik} \right) dx^k$$

or, $\delta \, A_i = A_{i,k} \, dx^k$ (5.9)

where $A_{i,k} = \dfrac{\partial A_i}{\partial x^k} - A_j \Gamma^j_{ik}$ (5.10)

Since δA_i are covariant components of a vector, and dx^k are contravariant components of any arbitrary vector hence we conclude from (5.9), by quotient rule, that $A_{i,k}$ are the covariant components of a second order tensor. We define $A_{i,k}$ as the covariant derivative of A_i.

Note : It should be noted that the covariant derivatives $A^i{}_{,k}$ and $A_{i,k}$ are different from the partial derivatives $\dfrac{\partial A^i}{\partial x^k}$ and $\dfrac{\partial A_i}{\partial x^k}$ respectively. In fact, the partial derivatives $\dfrac{\partial A^i}{\partial x^k}$ and $\dfrac{\partial A_i}{\partial x^k}$ are not the components of a tensor, in general. This may be shown by writing the law of their transformation to another system.

If the vector field under consideration is the cartesian coordinates, then the basis is constant and all Christoffel symbols $\Gamma_{ij,k}$ and Γ^k_{ij} vanish. In this case, the covariant derivatives of a function is identical to the partial derivatives of the function, i.e., when (x^i) is a cartesian system of coordinates then

$$A^i{}_{,k} = \frac{\partial A^i}{\partial x^k} \qquad and \qquad A_{i,k} = \frac{\partial A_i}{\partial x^k} \tag{5.11}$$

5.2.1 Covariant derivative of a scalar function :

Let ϕ define a scalar field of the coordinates x^k in the space; then $\frac{\partial \phi}{\partial x^k}$ are the covariant components of a vector.

Since $d\phi = \frac{\partial \phi}{\partial x^k} dx^k$, we see that the covariant derivative of ϕ is identical to the partial derivative of ϕ; that is

$$\phi_{,k} = \frac{\partial \phi}{\partial x^k} \tag{5.12}$$

5.2.2 Covariant derivative of relative scalar of weight :

Let $\tilde{\phi}$ be a relative scalar of weight ω, then $\left(\sqrt{g}\right)^{-\omega} \tilde{\phi} = \phi$ is an absolute scalar. If $\tilde{\phi}$ is a point function then

$$\frac{\partial \phi}{\partial x^k} = \frac{\partial}{\partial x^k}\left(g^{-\frac{\omega}{2}} \tilde{\phi} \right)$$

$$= g^{-\frac{\omega}{2}} \frac{\partial \tilde{\phi}}{\partial x^k} - \tilde{\phi} \frac{\omega}{2} g^{-\frac{\omega}{2}-1} \frac{\partial g}{\partial x^k}$$

$$= g^{-\frac{\omega}{2}} \left(\frac{\partial \tilde{\phi}}{\partial x^k} - \tilde{\phi}\, \omega \frac{1}{2g} \frac{\partial g}{\partial x^k} \right)$$

$$= g^{-\frac{\omega}{2}} \left(\frac{\partial \tilde{\phi}}{\partial x^k} - \tilde{\phi}\, \omega \frac{\partial}{\partial x^k} \left(\log \sqrt{g} \right) \right)$$

$$= g^{-\frac{\omega}{2}}\left(\frac{\partial \tilde{\phi}}{\partial x^k} - \tilde{\phi}\,\omega\,\Gamma^l_{kl}\right)$$

$$= \left(\sqrt{g}\right)^{-\omega}\tilde{\phi}_{,k}$$

where $\tilde{\phi}_{,k} = \frac{\partial \tilde{\phi}}{\partial x^k} - \tilde{\phi}\,\omega\,\Gamma^l_{kl}$ (5.13)

is called the covariant derivative of $\tilde{\phi}$ with respect to x^k. The $\tilde{\phi}_{,k}$ given by above equation is the covariant derivative of a relative scalar $\tilde{\phi}$ of weight ω.

5.2.3 Covariant derivative of a relative vector:

Let F^i be a relative vector of weight ω, then the vector f^i defined by

$$f^i = \left(\sqrt{g}\right)^{-\omega} F^i \qquad (5.14)$$

is an absolute vector. Now

$$\frac{\partial f^i}{\partial x^k} = \frac{\partial}{\partial x^k}\left(g^{-\frac{\omega}{2}} F^i\right)$$

$$\Rightarrow \frac{\partial f^i}{\partial x^k} = g^{-\frac{\omega}{2}}\frac{\partial F^i}{\partial x^k} - \frac{\omega}{2} g^{-\frac{\omega}{2}-1}\frac{\partial g}{\partial x^k} F^i$$

$$= g^{-\frac{\omega}{2}}\left(\frac{\partial F^i}{\partial x^k} - \omega\left(\frac{\partial \log\sqrt{g}}{\partial x^k}\right)F^i\right)$$

$$\frac{\partial f^i}{\partial x^k} = g^{-\frac{\omega}{2}}\left(\frac{\partial F^i}{\partial x^k} - \omega\Gamma^l_{kl}F^i\right) \qquad (5.15)$$

Multiplying (5.14) by Γ^i_{kj} and adding to (5.15) we get

$$\frac{\partial f^i}{\partial x^k} + \Gamma^i_{kj} f^j = g^{-\frac{\omega}{2}}\left[\frac{\partial F^i}{\partial x^k} + \Gamma^i_{kj}F^j - \omega\Gamma^l_{kl}F^i\right]$$

$$\Rightarrow f^i_{,k} = g^{-\frac{\omega}{2}}\left[\frac{\partial F^i}{\partial x^k} + \Gamma^i_{kj}F^j - \omega\Gamma^l_{kl}F^i\right]$$

$$\Rightarrow F^i_{,k} = \frac{\partial F^i}{\partial x^k} + \Gamma^i_{kj}F^j - \omega\Gamma^l_{kl}F^i \qquad (5.16)$$

which is a relative second-order mixed tensor of weight ω. Above formula defines the covariant derivative of a relative vector F^i of weight ω.

5.2.4 Covariant derivative of a second order tensors

(i) Let $\vec{\vec{T}}$ be a second-order tensor which can be expressed as

$$\vec{\vec{T}} = \vec{e}_i \vec{e}_j T^{ij} \tag{5.17}$$

The $d\vec{\vec{T}}$ differential of $\vec{\vec{T}}$ is given by

$$d\vec{\vec{T}} = \left(d\vec{e}_i\right)\vec{e}_j T^{ij} + \vec{e}_i\left(d\vec{e}_j\right)T^{ij} + \vec{e}_i\vec{e}_j\, dT^{ij}$$

$$= \left(\frac{\partial \vec{e}_i}{\partial x^k}\vec{e}_j T^{ij} + \vec{e}_i\frac{\partial \vec{e}_j}{\partial x^k}T^{ij} + \vec{e}_i\vec{e}_j\frac{\partial T^{ij}}{\partial x^k}\right)dx^k$$

$$= \left(\Gamma^p_{ik}\vec{e}_p\vec{e}_j T^{ij} + \Gamma^q_{jk}\vec{e}_q\vec{e}_i T^{ij} + \vec{e}_i\vec{e}_j\frac{\partial T^{ij}}{\partial x^k}\right)dx^k$$

$$= \left(\Gamma^i_{pk}\vec{e}_i\vec{e}_j T^{pj} + \Gamma^j_{qk}\vec{e}_j\vec{e}_i T^{iq} + \vec{e}_i.\vec{e}_j\frac{\partial T^{ij}}{\partial x^k}\right)dx^k$$

$$d\vec{\vec{T}} = \left[\left(\frac{\partial T^{ij}}{\partial x^k} + T^{pj}\Gamma^i_{pk} + T^{iq}\Gamma^j_{qk}\right)dx^k\right]\vec{e}_i\vec{e}_j \tag{5.18}$$

Let $$d\vec{\vec{T}} = \delta T^{ij}\vec{e}_i\vec{e}_j \tag{5.19}$$

then it follows that

$$\delta T^{ij} = \left(\frac{\partial T^{ij}}{\partial x^k} + T^{pj}\Gamma^i_{pk} + T^{iq}\Gamma^j_{qk}\right)dx^k$$

$$= T^{ij}_{,k}\, dx^k$$

where $$T^{ij}_{,k} = \frac{\partial T^{ij}}{\partial x^k} + T^{pj}\Gamma^i_{pk} + T^{iq}\Gamma^j_{qk} \tag{5.20}$$

which is called the covariant derivative of T^{ij}. $T^{ij}_{,k}$ are components of a tensor of order (2+1).

(ii) Let $\vec{\vec{T}}$ be a second order tensor which can be expressed as

$$\vec{\vec{T}} = \vec{e}^i, \vec{e}^j\, T_{ij} \tag{5.21}$$

The differential $d\vec{\vec{T}}$ of $\vec{\vec{T}}$ is given by

$$d\,\vec{\vec{T}} = \left(d\,\vec{e}^i\right)\vec{e}^j\,T_{ij} + \vec{e}^i\left(d\,\vec{e}^j\right)T_{ij} + \vec{e}^i\,\vec{e}^j\,dT_{ij}$$

$$= \left(\frac{\partial \vec{e}^i}{\partial x^k}\vec{e}^j\,T_{ij} + \vec{e}^i\,\frac{\partial \vec{e}^j}{\partial x^k}T_{ij} + \vec{e}^i\,\vec{e}^j\,\frac{\partial T_{ij}}{\partial x^k}\right)dx^k$$

$$= \left[-\Gamma^i_{pk}\vec{e}^p\,\vec{e}^j\,T_{ij} + \vec{e}^i\left(-\Gamma^j_{qk}\vec{e}^q\right)T_{ij} + \vec{e}^i\,\vec{e}^j\,\frac{\partial T_{ij}}{\partial x^k}\right]dx^k$$

$$= \left(-\Gamma^p_{ik}\vec{e}^i\,\vec{e}^j\,T_{pj} - \Gamma^q_{jk}\vec{e}^i\,\vec{e}^j\,T_{qi} + \vec{e}^i\,\vec{e}^j\,\frac{\partial T_{ij}}{\partial x^k}\right)dx^k$$

$$= \left[\left(\frac{\partial T_{ij}}{\partial x^k} - T_{pj}\Gamma^p_{ik} - T_{qi}\Gamma^q_{jk}\right)dx^k\right]\vec{e}^i\,\vec{e}^j$$

$$= \left[\left(\frac{\partial T_{ij}}{\partial x^k} - T_{pj}\Gamma^p_{ik} - T_{qi}\Gamma^q_{jk}\right)dx^k\right]\vec{e}^i\,\vec{e}^j \tag{5.22}$$

Let $$d\,\vec{\vec{T}} = \delta T_{ij}\,\vec{e}^i\,\vec{e}^j \tag{5.23}$$

then it follows that

$$\delta T_{ij} = \left(\frac{\partial T_{ij}}{\partial x^k} - T_{pj}\Gamma^q_{jk} - T_{qj}\Gamma^q_{ik}\right)dx^k$$

$$= T_{ij,k}\;dx^k$$

where $$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - T_{pi}\Gamma^p_{jk} - T_{qj}\Gamma^q_{jk} \tag{5.24}$$

which is called the covariant derivative of T_{ij}.$T_{ij,k}$ are components of a covariant tensor of order three.

(iii) Let $\vec{\vec{T}}$ be a tensor of second order. We express $\vec{\vec{T}}$ as

$$\vec{\vec{T}} = \vec{e}^{i}\,\vec{e}_{j}\,T_{i}^{j} \quad (5.25)$$

The differential $d\vec{\vec{T}}$ of $\vec{\vec{T}}$ is given by

$$d\vec{\vec{T}} = \left(d\vec{e}^{i}\right)\vec{e}_{j}\,T_{i}^{j} + \vec{e}^{i}\left(d\vec{e}_{j}\right)T_{i}^{j} + \vec{e}^{i}\,\vec{e}_{j}\,dT_{i}^{j}$$

$$= \left(\frac{\partial \vec{e}^{i}}{\partial x^{k}}\vec{e}_{j}\,T_{i}^{j} + \vec{e}^{i}\frac{\partial \vec{e}_{j}}{\partial x^{k}}T_{i}^{j} + \vec{e}^{i}\,\vec{e}_{j}\frac{\partial T_{i}^{j}}{\partial x^{k}}\right)dx^{k}$$

$$= \left(-\Gamma_{pk}^{i}\,\vec{e}^{p}\,\vec{e}_{j}\,T_{i}^{j} + \vec{e}^{i}\,\Gamma_{jk}^{q}\,\vec{e}_{q}\,T_{i}^{j} + \vec{e}^{i}\,\vec{e}_{j}\frac{\partial T_{i}^{j}}{\partial x^{k}}\right)dx^{k}$$

$$= \left(-\Gamma_{ik}^{q}\,\vec{e}^{i}\,\vec{e}_{j}\,T_{q}^{j} + \vec{e}^{i}\,\Gamma_{pk}^{j}\,\vec{e}_{j}\,T_{i}^{p} + \vec{e}^{i}\,\vec{e}_{j}\frac{\partial T_{i}^{j}}{\partial x^{k}}\right)dx^{k}$$

$$d\vec{\vec{T}} = \left[\left(\frac{\partial T_{i}^{j}}{\partial x^{k}} + T_{i}^{p}\,\Gamma_{pk}^{j} - T_{q}^{j}\,\Gamma_{ik}^{q}\right)dx^{k}\right]\vec{e}^{i}\,\vec{e}_{j} \quad (5.26)$$

Let $\quad d\vec{\vec{T}} = \delta T_{i}^{j}\,\vec{e}^{i}\,\vec{e}_{j} \quad (5.27)$

then if follows that

$$\delta\, T_{i}^{j} = \left(\frac{\partial T_{i}^{j}}{\partial x^{k}} + T_{i}^{p}\,\Gamma_{pk}^{j} - T_{q}^{j}\,\Gamma_{ik}^{q}\right)dx^{k}$$

where
$$T_{i,k}^{j} = \frac{\partial T_{i}^{j}}{\partial x^{k}} + T_{i}^{p}\,\Gamma_{pk}^{j} - T_{q}^{j}\,\Gamma_{ik}^{q} \quad (5.28)$$

which is called covariant derivative of T_{i}^{j}. $T_{i,k}^{j}$ are components of a mixed tensor of order (1+2)..

Theorem : The covariant derivatives of the base vectors $\vec{e}_{i}$, the reciprocal base vectors $\vec{e}^{i}$, the metric tensor g_{ij}, the inverse metric tensor g_{ij} and the $g = |g_{ij}|$ are all zero.

Proof : Since $\vec{e}_{i}$, $\vec{e}^{i}$, g_{ij} and g^{ij} are absolute tensors, therefore

$$\vec{e}_{i,j} = \frac{\partial \vec{e}_i}{\partial x^j} - \vec{e}_k \Gamma^k_{ij}$$

$$= \vec{e}_k \Gamma^k_{ij} - \vec{e}_k \Gamma^k_{ij}$$

$$\therefore \vec{e}_{i,j} = 0 \qquad (5.29a)$$

$$\vec{e}^i{}_{,j} = \frac{\partial \vec{e}^i}{\partial x^j} + \vec{e}^k \Gamma^i_{jk}$$

$$= -\vec{e}^k \Gamma^i_{jk} + \vec{e} \Gamma^i_{jk}$$

$$\therefore \vec{e}^i{}_{,j} = 0 \qquad (5.29b)$$

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - g_{pj} \Gamma^p_{ik} - g_{qi} \Gamma^q_{jk}$$

$$= \frac{\partial g_{ij}}{\partial x^k} - \Gamma_{ik,j} - \Gamma_{jk,i}$$

$$= \Gamma_{ik,j} + \Gamma_{jk,i} - \Gamma_{ik,j} - \Gamma_{jk,i}$$

$$\therefore g_{ij,k} = 0 \qquad (5.30a)$$

$$g^{ij}_{,k} = \frac{\partial g^{ij}}{\partial x^k} + g^{pj} \Gamma^i_{pk} + g^{qi} \Gamma^j_{qk}$$

$$= -g^{pj} \Gamma^i_{pk} - g^{qi} \Gamma^j_{qk} + g^{pj} \Gamma^i_{pk} + g^{qi} \Gamma^j_{qk} \qquad \text{(by 4.21)}$$

$$\therefore g^{ij}_{,k} = 0 \qquad (5.30b)$$

Since $g = |g_{ij}|$ is a relative scalar of weight two, therefore

$$g_{,k} = \frac{\partial g}{\partial x^k} - 2g \Gamma^i_{ik}$$

$$= 2g \left[\frac{1}{2g} \frac{\partial g}{\partial x^k} - \Gamma^i_{ik} \right]$$

$$= 2g \left[\frac{\partial}{\partial x^k} \left(\log \sqrt{g} \right) - \frac{\partial}{\partial x^k} \left(\log \sqrt{g} \right) \right]$$

$$\therefore g_{,k} = 0 \qquad (5.31)$$

As a consequence of constancy of g_{ij} and g^{ij} with respect to covariant differentiation, we have

$$A^i_{,k} = \left(g^{ij} A_j\right)_{,k} = g^{ij} A_{j,k} \tag{5.32a}$$

$$A_{i,k} = \left(g_{ij} A^j\right)_{,k} = g_{ij} A^j_{,k} \tag{5.23b}$$

But this is precisely what one expects if the metric tensors are to be used for raising and lowering indices of all tensors, including those arise from covariant differentiation.

5.3 Rules of Covariant Derivatives:

Since covariant differentiation is directly related to the partial differentiation, the rules of partial differentiation are applicable to the covariant differentiation as well.

Theorem: If A^i and B^i are two vectors then

$$\left(A^i + B^i\right)_{,k} = A^i_{,k} + B^i_{,k}$$

Proof: Since A^i and B^i are two vectors therefore $A^i + B^i$ is also a vector. Now

$$\left(A^i + B^i\right)_{,k} = \frac{\partial}{\partial x^k}\left(A^i + B^i\right) + \left(A^j + B^j\right)\Gamma^i_{kj}$$

$$= \frac{\partial A^i}{\partial x^k} + \frac{\partial B^i}{\partial x^k} + A^j \Gamma^i_{kj} + B^j \Gamma^i_{kj}$$

$$= \left(\frac{\partial A^i}{\partial x^k} + A^j \Gamma^i_{kj}\right) + \left(\frac{\partial B^i}{\partial x^k} + B^j \Gamma^i_{kj}\right)$$

$$\therefore \left(A^i + B^i\right)_{,k} = A^i_{,k} + B^i_{,k} \tag{5.33}$$

This shows that covariant derivative of the sum of two vectors A^i and B^i is equal to the sum of the covariant derivatives of A^i and B^i.

Theorem : If A^i and B_j are two vectors then

$$\left(A^i + B_j\right)_{,k} = A^i_{,k} + A^i B_{j,k}$$

Proof : If A^i is a contravariant vector and B_j is a covariant vector then we know tht their outer product A^iB_j is a mixed thensor of order two.

$$\left(A^iB_j\right)_{,k} = \frac{\partial}{\partial x^k}\left(A^iB_j\right) + A^pB_j\Gamma^i_{pk} - A^iB_q\Gamma^q_{jk}$$

$$= \frac{\partial A^i}{\partial x^k}B_j + A^i\frac{\partial B_j}{\partial x^k} + A^pB_j\Gamma^i_{pk} - A^iB_q\Gamma^q_{jk}$$

$$= \left(\frac{\partial A^i}{\partial x^k} + A^p\Gamma^i_{pk}\right)B_j + A^i\left(\frac{\partial B_j}{\partial x^k} - B_q\Gamma^q_{jk}\right)$$

$$\therefore \left(A^iB_j\right)_{,k} = A^i_{,k}\,B_j + A^iB_{j,k} \qquad (5.34)$$

Theorem: The order of operation of contraction and covariant differentiation is interchangeable.

Proof: For a tensor A^i_j we have

$$A^i_{j,k} = \frac{\partial A^i_j}{\partial x^k} + A^p_j\Gamma^i_{pk} - A^i_q\Gamma^q_{jk}$$

Equating $i = j$, we get

$$A^i_{i,k} = \frac{\partial A^i_i}{\partial x^k} + A^p_i\Gamma^i_{pk} - A^i_q\Gamma^q_{ik}$$

$$= \frac{\partial A^i_i}{\partial x^k} + A^i_q\Gamma^q_{ik} - A^i_q\Gamma^q_{ik}$$

$$= \frac{\partial A^i_i}{\partial x^k}$$

$$= \left(A^i_i\right)_{,k} \qquad (5.35)$$

5.4 Gradient, Divergence, Curl and Laplaciam in tensor notation.

Gradient : Let ϕ be an absolute scalar function, then

$$d\phi = \left(dx^i\,\vec{e}_i\right).\nabla\phi$$

we also have

$$d\phi = \phi_{,i}dx^i$$

$$\therefore \vec{e}_i.\nabla\phi = \phi_{,i}$$

$$\Rightarrow \nabla\phi = \vec{e}^{\,i}\,\phi_{,i} \qquad (5.36)$$

Above expression defines the gradient of a scalar ϕ where $\phi_{,i}$ is covariant derivative of ϕ and reduces to the partial derivative $\frac{\partial \phi}{\partial x^i}$.

Divergence : If $\vec{A} = A^i \vec{e}_i$ then divergence of $\vec{A}$ is defined as the contraction of covariant derivative of A^i with respect to x^i. Thus

$$\nabla . \vec{A} = A^i_{,i} \tag{5.37}$$

Theorem : $\nabla . \vec{A} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g}\, A^i \right)$

Proof :

$$\nabla . \vec{A} = A^i_{,i} = \frac{\partial A^i}{\partial x^i} + A^p \Gamma^i_{pi}$$

$$= \frac{\partial A^i}{\partial x^i} + A^p \frac{\partial}{\partial x^p} \left(\log \sqrt{g} \right)$$

$$= \frac{\partial A^i}{\partial x^i} + A^p \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^p}$$

$$= \frac{\partial A^i}{\partial x^i} + A^i \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial x^i}$$

$$= \frac{1}{\sqrt{g}} \left(\frac{\sqrt{g} \partial A^i}{\partial x^i} + A^i \frac{\partial \sqrt{g}}{\partial x^i} \right)$$

$$\nabla . \vec{A} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} A^i \right) \tag{5.38}$$

Curl of a vector. The curl of a vector $\vec{A}$ is defined defined by

$$\text{curl } \vec{A} = \left(\varepsilon^{ijk} A_{k,j} \right) \vec{e}_i \tag{5.39}$$

Laplacian: The Laplacian of a scalar function ϕ is defined as the divergence of grad ϕ i.e.

$$\nabla^2 \phi = div \left(\vec{e}^{\,i} \phi_{,i} \right) \tag{5.40}$$

Theorem: $\nabla^2 \phi = \dfrac{1}{\sqrt{g}} \dfrac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{jk} \dfrac{\partial \phi}{\partial x^k} \right)$

Proof: $\nabla^2 \phi = \text{div}\left(\text{grad}\, \phi\right)$

$$= div\left(\vec{e}^{\,k} \phi_{,k} \right).$$

$$= \text{div}\left(g^{jk} \phi_{,k} \vec{e}_j \right)$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{jk} \phi_{,k} \right) \qquad \text{(using 5.38)}$$

$$\therefore \nabla^2 \phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j}\left(\sqrt{g}\, g^{jk} \frac{\partial \phi}{\partial x^k} \right) \tag{5.41}$$

5.5 The Riemann - Christoffel tensor

In case of tensor the order of covariant differentiation is not, in general, commutative. Let us discuss under what conditions can we write

$$A_{i,jk} = A_{i,kj} \tag{5.42}$$

$$A_{i,jk} = \left(A_{i,j} \right)_{,k}$$

$$= \frac{\partial A_{i,j}}{\partial x^k} - A_{m,j} \Gamma^m_{ik} - A_{i,m} \Gamma^m_{jk}$$

$$= \frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j} - A_m \Gamma^m_{ij} \right) - \left(\frac{\partial A_m}{\partial x^j} - A_n \Gamma^n_{mj} \right)\Gamma^m_{ik} - \left(\frac{\partial A_i}{\partial x^m} - A_n \Gamma^n_{im} \right)\Gamma^m_{jk}$$

$$A_{i,jk} = \frac{\partial^2 A_i}{\partial x^k \partial x^j} - A_m \frac{\partial}{\partial x^k}\Gamma^m_{ij} - \Gamma^m_{ij} \frac{\partial A_m}{\partial x^k} - \Gamma^m_{ik} \frac{\partial A_m}{\partial x^j}$$

$$+ \Gamma^n_{mj} \Gamma^m_{ik} A_n - \Gamma^m_{jk} \frac{\partial A_i}{\partial x^m} + \Gamma^n_{im} \Gamma^m_{jk} A_n \tag{5.43}$$

Interchaning *j* and *k* in (5.43), we get

$$A_{i,kj} = \frac{\partial^2 A_i}{\partial x^j \partial x^k} - A_m \frac{\partial}{\partial x^j}\Gamma^m_{ik} - \Gamma^m_{ik} \frac{\partial A_m}{\partial x^j} - \Gamma^m_{ij} \frac{\partial A_m}{\partial x^k}$$

$$+ \Gamma^n_{mk} \Gamma^m_{ij} A_n - \Gamma^m_{jk} \frac{\partial A_i}{\partial x^m} + \Gamma^n_{im} \Gamma^m_{jk} A_n \tag{5.44}$$

Hence,

$$A_{i,jk} - A_{i,kj} = A_m \frac{\partial}{\partial x^j}\Gamma^m_{ik} - A_m \frac{\partial}{\partial x^k}\Gamma^m_{ij}$$

$$+\Gamma^m_{ik}\Gamma^n_{mj}A_n - \Gamma^m_{ij}\Gamma^n_{mk}A_n$$

$$= A_m\left[\frac{\partial}{\partial x^j}\Gamma^m_{ik} - \frac{\partial}{\partial x^k}\Gamma^m_{ij} + \Gamma^m_{nj}\Gamma^n_{ik} - \Gamma^m_{nk}\Gamma^n_{ij}\right]$$

Thus we write

$$A_{i,jk} - A_{i,kj} = A_m R^m_{ijk} \tag{5.45}$$

where

$$R^m_{ijk} = \frac{\partial}{\partial x^j}\Gamma^m_{ik} - \frac{\partial}{\partial x^k}\Gamma^m_{ij} + \Gamma^m_{nj}\Gamma^n_{ik} - \Gamma^m_{nk}\Gamma^n_{ij} \tag{5.46}$$

is the Riemann Christoffel thensor and the symbols R^m_{ijk} are called Riemann's symbols of the second kind. From (5.46) it follows that

$$R^m_{ijk} = -R^m_{ikj} \tag{5.47}$$

we see from (5.45) that the vanishing of the Riemann Christoffel tensor is a necessary and sufficient condition for mixed covariant derivatives (5.42). We do make a final note here that the R-C tensor is a measure of the 'curvature' of a particular space and its vanishing is a condition for the space to be Euclidean i.e., the space to be "flat".

5.6 Covariant curvature tensor

The tensor $R^n_{n,ijk} = g_{nm} R^m_{ijk}$ (5.48)

is an associate tensor of the tensor R^m_{ijk} and is a covariant tensor of rank four. This tensor is called the covariant curvature tensor and the symbols R_{nijk} are called Riemann's symbols of the first kind. Substituting the value of R^m_{ijk} from (5.46) in (5.48) and simplifying, we get the following expression for the tensor R_{nijk}:

$$R_{nijk} = \frac{\partial}{\partial x^j}\Gamma_{ik,n} - \frac{\partial}{\partial x^k}\Gamma_{ij,n} + \Gamma_{nk,m}\Gamma^m_{ij} - \Gamma_{nj,m}\Gamma^m_{ik} \tag{5.49}$$

On putting the values of the Christoffel symbols of the first kind, we have the following alternative expression for the covariant curvature tensor

$$R_{nijk} = \frac{1}{2}\left[\frac{\partial^2 g_{nk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^n \partial x^k} - \frac{\partial^2 g_{nj}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^n \partial x^j}\right]$$

$$+ g_{ms}\Gamma^s_{mk}\Gamma^m_{ij} - g_{ms}\Gamma^s_{nj}\Gamma^m_{ik} \qquad (5.50)$$

From this expression, it follows that

$$R_{nijk} = - R_{injk} \qquad (5.51)$$

$$R_{nijk} = - R_{nikj} \qquad (5.52)$$

$$R_{nijk} = R_{jkni} \qquad (5.53)$$

and

$$R_{iijk} = R_{nijj} = 0 \qquad (5.54)$$

The first two equations (5.51) and (5.52) show that the covariant curvature tensor is skew-symmetric in the first two indices *n* and *i* as well as in the last two indices *j* and *k*.

5.7 Examples:

5.1. Prove that $\delta^i_{j,k} = 0$

Solution: $\delta^i_{j,k} = \dfrac{\partial \delta^i_j}{\partial x^k} + \delta^l_j \Gamma^i_{lk} - \delta^i_l \Gamma^l_{jk}$

$$= 0 + \Gamma^i_{jk} - \Gamma^i_{jk}$$

$$= 0$$

5.2. Prove that

$$\left(g_{ij} A^{jp}_q\right)_{,k} = g_{ij} A^{jp}_{q,k}$$

Solution: By the product rule

$$\left(g_{ij} A^{jp}_q\right)_{,k} = g_{ij,k} A^{jp}_q + g_{ij} A^{jp}_{q,k}$$

$$= 0 + g_{ij} A^{jp}_{q,k}$$

$$= g_{ij} A^{jp}_{q,k}$$

5.3. Prove that $\text{curl } A_m = \dfrac{\partial A_m}{\partial x^n} - \dfrac{\partial A_n}{\partial x^m}$

Solution: $\text{curl } A_m = A_{m,n} - A_{n,m}$

$$= \frac{\partial A_m}{\partial x^n} - A_s \Gamma^s_{mn} - \frac{\partial A_n}{\partial x^m} + A_s \Gamma^s_{nm}$$

$$= \frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m}.$$

5.4. Express the divergence of a vector $\vec{A}$ in terms of its physical components for cylindrical coordinates.

Solution : For cylindrical coordinates

$$x^1 = r,\; x^2 = \theta,\; x^3 = z$$

$$g = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = r^2 \quad \text{and} \quad \sqrt{g} = r$$

The physical components denoted by $A(r), A(\theta), A(z)$ are given by

$$A(r) = \sqrt{g_{11}}\, A^1 = A^1, \quad A(\theta) = \sqrt{g_{22}}\; A^2 = rA^2$$

$$A(z) = \sqrt{g_{33}} = A^3 = A^3.$$

Therefore,

$$\text{div } \vec{A} = A^k_{,k} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k}\left(\sqrt{g} A^k\right)$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA(r)) + \frac{\partial}{\partial \theta}(A(\theta)) + \frac{\partial}{\partial z}(A(z))\right].$$

5.5. Prove that

$$R^m_{ijk} + R^m_{jki} + R^m_{kij} = 0$$

Solution : Since

$$R^m_{ijk} = \frac{\partial}{\partial x^j}\Gamma^m_{ik} - \frac{\partial}{\partial x^k}\Gamma^m_{ij} + \Gamma^m_{nj}\Gamma^n_{ik} - \Gamma^n_{nk}\Gamma^n_{ij}$$

Cyclic permutation of indices i, j, k gives two more equations:

$$R^m_{jki} = \frac{\partial}{\partial x^k}\Gamma^m_{ji} - \frac{\partial}{\partial x^i}\Gamma^m_{jk} + \Gamma^m_{nk}\Gamma^n_{ji} - \Gamma^m_{ni}\Gamma^n_{jk}$$

and

$$R^m_{kij} = \frac{\partial}{\partial x^i}\Gamma^m_{kj} - \frac{\partial}{\partial x^j}\Gamma^m_{ki} + \Gamma^m_{ni}\Gamma^n_{kj} - \Gamma^m_{nj}\Gamma^m_{ki}$$

Since the Christoffel symbols Γ^m_{ij} are symmetric in i and j, therefore adding above three expressions, we get

$$R^m_{ijk} + R^m_{jki} + R^m_{kij} = 0$$

5.8 Exercise:

5.1. Show that the covariant derivatives of ε^{ijk}, ε_{ijk} and $\varepsilon^{ijk}\,\varepsilon_{lmn}$ vanish.

5.2. In orthogonal coordinate system, prove that

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_3h_1}{h_2}\frac{\partial\phi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial x^3}\right)\right]$$

5.3. In orthogonal coordinate system, prove that

$$\text{div}\,\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\{A(1)h_2h_3\} + \frac{\partial}{\partial x^2}\{A(2)h_3h_1\} + \frac{\partial}{\partial x^3}\{A(3)h_1h_2\}\right]$$

5.4. Prove that

$$\left(A^i_jB_k\right)_{,p} = A^i_{j,p}B_k + A^i_jB_{k,p}$$

5.5. Show that

$$R_{pijk} + R_{pjki} + R_{pkij} = 0$$

5.9 Let Us Sum Up

In this block, you learnt about covariant differentiation of various order of tensors. You have also learnt how to find the expressions of gradient, divergence, curl and Laplacian in various coordinate systems. You have also defined the Riemann-Christoffel symbol.

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