SELF LEARNING MATERIAL

Master of Arts/Science

MATHEMATICS

COURSE: MATH - 201

COMPLEX ANALYSIS

BLOCK : 1, 2 & 3

DIRECTORATE OF OPEN AND DISTANCE LEARNING
DIBRUGARH UNIVERSITY
DIBRUGARH-786004

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Published on behalf of the Directorate of Open & Distance Learning, Dibrugarh University by the Director, DODL, D.U. and printed at SUNPRINT, Dibrugarh-04.

MATHEMATICS

COURSE : MATH - 201

COMPLEX ANALYSIS

BLOCK-1: COMPLEX INTEGRATION [16 Marks]

STRUCTURE

1.0 **Objectives**

- ❖ After going through this block will be able to
- ❖ Intergrate complex-valued function of a real variable
- ❖ Obtain Use contour Integration
- ❖ Use Cauchy-Goursat theorem to find some particular integral.

When the individual integrals on the right exist.
Thus we have

$$\operatorname{Re}\int_a^b w(t)\,dt = \int_a^b u(t)\,dt = i\int_a^b \operatorname{Re} w(t)\,dt$$

and $$\operatorname{Im}\int_a^b w(t)\,dt = \int_a^b v(t)\,dt = i\int_a^b \operatorname{Im} w(t)\,dt$$

Consider fthe following example to have a better idea :

Example 1 : $\int_0^1 (1+it)2dt = \int_0^1 (1-t2)\,dt + i\int_a^b 2tdt = 2/3+i$

By applying the fundamental theorem of calculus to real and imaginary part, we observe that it is also applicable to the integral $\int_a^b w(t)\,dt$.

Example 2 : $\int_0^{\pi/4} e^{it}dt = -ie^{it}$

Improper integrals of $w(t)$ over unbounded intervals are defined in a similar way. that is

$$\int_a^{\propto} f(z)\,dz = w(t)\,dt \text{ as } R \to \propto$$

The existance of the integrals of *u* and *v* is ensured if these functions are continuous piecewise on [*a,b*]. A function is said to be piecewise continuous on [*a,b*] if [*a,b*] can be broken up into finite number of subintervals in which the function in continuous.

❑ *Point to remember* : To evaluate integral of any complex valued function of real variable $w(t)=u(t)+iv(t)$, simply apply the rules of integration of real functions to u(t) and v(t).

Example : $f(z)=1/z$ has isolated singularity at z=0, whereas $f(z)=1/\sin\pi/z$ has singularities at z=0, 1/n (n=1, 2, 3, ..., n) of which z=0 is a non isolated singularity.

Result : If f is continuous and non-zero at a point a then $f(z)\neq 0$ throughout some neighbourhood of that point.

Note: For $f=u+iv$ to be analytic at a point a is that u and v possesses continuous first order partial derivative and satisfy CauchyRiemann's equation at that point, i.e. u_x, u_y, v_x, v_y are continuous and satisfy $u_x=v_y$ and $u_y=-v_x$.

Definition (Harmonic function) : A real valued function u(x,y) of two real veriable is said to be harmonic in a domain D, if the second order partial derivatives of u are continuous and satisfy the Laplace equation $u_{xx}+u_{yy}=0$.

Definition (Harmonic conjugate) : Let u be harmonic in a domain D. If there exixts a real valued function v(x,y) of two variable such that u+iv is analytic in D, then v is said to be harmonic conjugate of u.

Note : If v is a harmonic conjugate of u, then for any constant c, *v*=*c* is also a harmonic conjugate of u.

Result : For any harmonic function u in a domain D there always exist a harmonic conjugate.

Theorem (Green's theorem) : If *D* is a domain bounded by a closed contour *C*, and *u* and *v* are real valued functions of two real variables, whose partial derivatives are continuous on *d*, then

$$\iint_D \left(v_y-u_x\right)dxdy=\int_C \left(udx+vdy\right)$$

Definition **(Arc)**: Consider a continuous function from some interval [*a,b*] to *C*, the set points of the image is ordered by the natural ordering of the reals of [*a,b*]. this set is called an arc

Note : In our study, unless otherwise mentioned, positive direction of closed contour will be considered.

1.3.2 **Complex Integration**

Let $f : C \to C$ be a complex function. Then $f = u + iv$, when *u* and *v* are complex-valued functions of two real variable. Let *C* be a contour given by $z : [a,b] \to C$, where $z(t) = x(t) + iy(t)$. Then we define the line integral, or contour integral of *f* along *C* as

$$\int_c f(z)dz = \int_a^b f(z(t))z'(t)dt$$

The RHS is the integral of a complex valued function of real variable over [a, b]. So all the results of section 4.2 are applicable here.

Example 1 : $f(z) = z - 1$ and C is the arc from *z*=0 to *z*=2 consisting of the segment $0 \le x \le 2$

Here parametric representation of *C* is

$$z(t) = t - i0\,,\ 0 \le t \le 2$$

$$\therefore \int_c f(z)dz = \int_0^2 f(z(t))z'(t)dt = \int_0^2 f(z(t) - 1)dt$$

$$= \int_0^2 f(t-1)dt = 0$$

Example 2 **:** Let $f(z) = \bar{z}$ and C be given by

$$z(t) = 2e^{it}, \qquad -\pi/2 \le t \le \pi/2$$

then $\int_c f(z)dz = \int_{-\pi/2}^{\pi/2} f(z(t))z'(t)dt$

$$\int_{-\pi/2}^{\pi/2} \bar{z}(t)2ie^{it}dt = \int_{-\pi/2}^{\pi/2} 2e^{-it}.2ie^{it}dt$$

$$= \int_{-\pi/2}^{\pi/2} dt = 4\pi i$$

$$= \int_{-b}^{-a} f\left(z(-t)\right) z'(-t)\, dt$$

Put s=-t then ds= -dt

$$\therefore \int_{-c} f\left(z_1\right) dz_1 = \int_{b}^{a} f\left(z(s)\right) z'(s)\, ds$$

i.e. $$\int_{-c} f\left(z_1\right) dz_1 = -\int_{b}^{a} f\left(z(s)\right) z'(s)\, ds$$

$$\Rightarrow \int_{-c} f(z)\,dz = -\int_{c} f(z)\,dz$$

(d) Let C$_1$ be given by $z_1 : [a,b] \to C$

C$_2$ be given by $z_2 : [b,d] \to C$

C be given by $z_3 : [a,b] \to C : z(t) = z_1(t), a \le t \le b$

$$= z_2(t), \quad b \le t \le d$$

Then

$$\int_{c} f(z)\, dz = \int_{a}^{d} f\left(z(t)\right) z'(t)\, dt$$

But $f\left(z(t)\right) z'(t)$ is a complex valued funfction defined on [*a, b*]. So we can write

$$\int_{a}^{d} f\left(z(t)\right) z'(t)\, dt = \int_{a}^{b} f\left(z(t)\right) z'(t)\, dt + \int_{b}^{d} f\left(z(t)\right) z'(t)\, dt$$

$$= \int_{a}^{b} f\left(z_1(t)\right) z_1'(t)\, dt + \int_{b}^{d} f\left(z_2(t)\right) z_2'(t)\, dt$$

$$= \int_{C_1} f\left(z_1\right) dz_1 + \int_{C_2} f\left(z_2\right) dz_2$$

$$= \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz$$

(e) We have

$$\left| \int_{c_1} f(z)\, dz \right| = \left| \int_{a}^{b} f\left(z(t)\right) z'(t)\, dt \right| \le \int_{a}^{b} \left| f\left(z(t)\right) \right| \left| z'(t) \right| dt \le M \int_{a}^{b} \left| z'(t) \right| dt,$$

$$=\int_0^1\left[x(t)+y(t)-3i(x(t))^3\right]dt+\int_0^1\left[x(t)+y(t)-3i(xt)^3\right]idt$$

$$=\int_0^1\left[t-3it^3\right]dt+i\int_0^1 tdt$$

$$=\frac{5-3i}{2}$$

Example 2 : Let C_1 and C_2 be as in example 1 and let $g(z)=x+(y+2)i$

Then

$$\int_{c_1} g(z)\,dz=\int_0^1\left[x(t)+(y(t)+2)i\right](1+i)\,dt$$

$$=\int_0^1\left[t+(t+2)i\right](1+i)\,dt=-2+3i$$

and $$\int_{c_1} g(z)\,dz=\int_{OA} g(z_1)\,dz_1=\int_{AB} g(z_1)\,dz_1$$

$$=\int_0^1(x(t)+2i)dt+\int_0^1(y(t)+2)i\;idt$$

$$=\int_0^1(t+2i)dt-\int_0^1(t+2)\,dt=-2+3i$$

Consider the following example

Example 3 : Let C be the contour given by $z:[o,2\pi]\to C$, $z(t)=\cos t+i\sin t=e^{it}$

Let $f(\text{z})=\text{z}^2$

Then

$$\int_C f(z)\,dz=\int_0^{2\pi}(z(t))^2(-\sin t+i\cos t)\,dt$$

$$=\int_0^{2\pi}\left[(x(t))^2-\left(y(t)^2\right)+2ix(t)\,y(t)\right](-\sin t+i\cos t)\,dt$$

Theorem 1: Let f be continuous in a domain D. Then the following statements are equivalent.

(a) f has antiderivative F in D.

(b) Integrals of $f(z)$ between two fixed points in D is independent of path.

(c) The integral of $f(z)$ along every closed contour in D is zero.

Proof : $(a) \Rightarrow (b)$. Let C be a smooth curve given by $z(t) = x(t) + iy(t) \quad (a \le t \le b)$ such that $z(a) = z_1$ and $z(b) = z_2$.

Given $F'(z) = f(z), \; \forall z \in D$

Then by definition

$$\int_c f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b F'(z(t))\,z'(t)\,dt$$

Since fundamental theorem of calculus can be extended to complex-valued function of a real variables we have

$$\int_c f(z)\,dz = \left[F(z(t))\right]_a^b = F(z(b)) - F(z(a)) = F(z_2) - F(z_1)$$

Then the value of the integral is independent of the path.

If C is a contour made of finite number of smooth curves $C_1, C_2, \ldots\ldots\ldots\ldots C_n$ joined end to end such that each C_n connects z_n to z_{n+1}. Then

$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \ldots\ldots + \int_{c_n} f(z)\,dz$$

$$= \left[F(z_2) - F(z_1)\right] + \left[F(z_3) - F(z_2)\right] + \ldots + \left[F(z_n) - F(z_{n-1})\right]$$

$$= F(z_n) - F(z_1)$$

which is again independent of the path.

$(b) \Leftrightarrow (c)$

Let C_1 and C_2 be two contour in D connecting two fixed points z_1 and z_2.

Since $\int_{z_1}^{z+\Delta z} ds = \Delta z$

we have $f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,dz$

Then

$$\frac{F(z+\Delta z)-F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(s)\,ds - f(z)$$

$$= \frac{1}{\Delta z}\int_{z}^{z+\Delta z}\left[f(s)-f(z)\right]ds$$

As f is continuous at z for $\in > 0$, $\exists \delta > 0$:

$$\left|f(s)-f(z)\right| < \in \text{ whenever } \left|s-z\right| < \delta$$

Take $\left|\Delta z\right| < \delta$. Then $\left|f(s)-f(z)\right| < \in$

So,

$$\left|\frac{F(z+\Delta z)-f(z)}{\Delta z} - f(z)\right| < \frac{1}{\left|\Delta z\right|} \in \left|\Delta z\right| = \in \text{ whenever } \left|\Delta z\right| < \delta$$

i.e. $\underset{\Delta z \to 0}{Lt}\frac{F(z+\Delta z)-f(z)}{\Delta z} = f(z)$, is $F'(z) = f(z)$, $\quad \forall z \in D$

vvYou should note that the above theorem does not claim that any of the statements is true for a given function and a given domain D. All it claims is that either all of them are true or none of them is true.

Example 1 : The continuous function $f(z) = z^3$ has antiderivative $F(z) = z^4/4$ throughout the complex plane. Hence by the above theorem, integral between two points is independent of path.

1.4.2 **Cauchy theorem :**

In the previous section we have seen that when a continuous function f has an antiderivative in a domain *D*, the integral of $f(z)$ around any given closed contour lying in *D* is zero. In this section we will deal with a theorem which give other conditions on the function f, which ensure that the value of the integral of $f(z)$ around a simple closed contour is zero.

***Theorem 1* (Cauchy theorem) :** If *f* is analytic and *f*' is continuous within and on a simple cloud contour C, then

$$\int_c f(z)\,dz = 0$$

Proof : Let *C* be given by the parametric representation $z(t) = x(t) + iy(t), \quad 0 \le t \le b$

Then by the definition

$$\int_c f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

$$= \int_a^b \left[u(x(t), y(t)) + iv(x(t), y(t))\right]\left[x'(t) + iy'(t)\right]dt$$

$$= \int_a^b (ux' - vy')\,dt + i\int_a^b (vx' + uy)\,dt$$

which can be written as

$$\int_c f(z)\,dz = \int_c (udx - vdy) + i\int_c (vdx + udy) \quad \text{..................} \text{ (i)}$$

we will express the line integrals on the RHS as double integral using Green's theorem. As *f*' is continuous, the first order partial derivatives of u and v are also continuous within *C*. Hence by Green's theorem

$$\int_c udx - \mathrm{v}dy = \iint_D (-v_x - u_y)\,dxdy$$

it encloses only points of D is called a simply connected domain. Geometrically it mean that every simple closed contour in D can be shrunk to a point without going outside the domain. A domain that is not simply connected is called multiply connected.

The interior of a simple closed contour and a set of interior points of a circle or ellipse or rectangle are simply connected domains. The infinite strip $1 < \operatorname{Re} z < 2$ is another example of a simply connected dfomain.

The annular domain $\{z : 1 < |z| < 2\}$ is an example of a multiply connected domain.

The following theorem is an extension of Cauchy Gaursal theorem to simply connected domain theorem. If f is analytic throughout a simply connected domain D, then

$$\int_c f(z)\, dz = 0$$

for every simple closed contour C in D.

This given the following corollary.

Corollary : If f is analytic throughout a simply connected domain D, then f must have antiderivative in D.

Proof : Since f is analytic throughout a simply connected domain D. we have by the previous theorem

$$\int_c f(z)\, dz = 0$$

for every simple closed contour C lying in D . Again we know from theorem that this implies f has antiderivative in D.

Remark : An entire function is analytic throughout the complex plane (which is simply connected). Hence by the corollary any entire function possesses antiderivative.

we have

$$\int_{c} f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz = 0$$

Hence C, C_1 , C_2 are taken with positive direction. If you can find $\int_{c_1} f(z)\,dz$ and $\int_{c_2} f(z)\,dz$, than $\int_{C} f(z)\,dz$ is known

Example 2 : Let $C_1 : |z| = 4$ and C_2 be the boundary of the square with sides $x = \pm 1, y = \pm 1$. then

$$\int_{C_1} \frac{1}{3z^2+1}\,dz = \int_{C_2} \frac{dz}{3z^2+1}$$

since the integral is analytic on C_1 and C_2 and in the domain enclosed by these contour.

> **Check your progress**
>
> (d) Verify example 2, by evaluating both the integrals.

Exercise : Use the principle of deformation to show that

$$\int_{C} \frac{dz}{(z-a)^n} = \begin{cases} 2\pi i, n = 1 \\ 0, n \neq 1 \end{cases}$$

where C is any positively oriented simple closed contour enclosing a.

Solution : Let $c_1 : |z-a| = r$, i.e. $z = a - re^{it}$, $0 \le t \le 2\pi$.

As 'a' is the only singularflity of the integral, by principle of diformation of paths, we have

$$\int_{c} \frac{dz}{(z-a)^n} = \int_{c_1} \frac{dz}{(z-a)^n}$$

For n=1,

$$\int_{c_1} \frac{dz}{(z-a)^n} = \int_{0}^{2\pi} \frac{z'(t)\,dt}{z(t)-a}$$

$$\int_C \frac{f(z)dz}{z-a} = \int_{C_r} \frac{f(z)dz}{z-a}$$

Then

$$\int_C \frac{f(z)dz}{z-a} - f(a)\int_{C_r} \frac{f(z)dz}{z-a} = \int_{C_r} \frac{f(z)-f(a)}{z-a}dz \quad \text{....... (i)}$$

We have already shown

$$\int_{c_r} \frac{f(z)dz}{z-a} = 2\pi i$$

Again *f* being analytic, is continuous at a. So for $\in > 0$, $\exists \delta > 0$:

$$|f(z)-f(a)| < \in \text{ whenever } |z-a| < \delta$$

we choose c_r such that $|z-a| < \delta$, $\forall z \in c_r$

So,

$$\left|\int_{C_r} \frac{f(z)-f(a)}{z-a}dz\right| \le \int_{C_r}\left|\frac{f(z)-f(a)}{z-a}\right|dz < \in/r \int_{C_r} dz = \in/r \, 2\pi r$$

Then (i) $\Rightarrow$

$$\left|\int_C \frac{f(z)dz}{z-a} - 2\pi i f(a)\right| = \left|\int_{C_r} \frac{f(z)-f(a)}{z-a}dz\right| < 2\pi \in$$

we can choose $\in > 0$ arbitrarily small, hence we conclude

$$\int_C \frac{f(z)dz}{z-a} - 2\pi i f(a)$$

i.e. $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)dz}{z-a}$

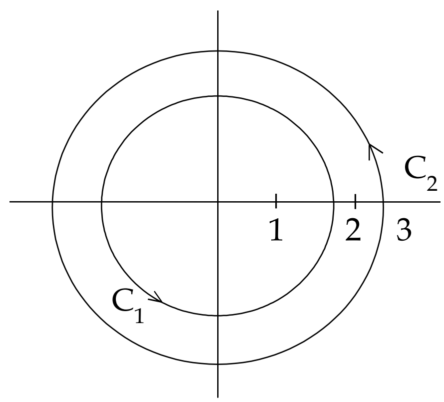
Example : Consider the integral

$$\int_{|z|=1} \frac{\cos\ 2\pi z}{(2z-1)(z-3)}dz$$

Let $f(z) = \dfrac{\cos\ 2\pi z}{2(z-3)}$. Then *f* is analytic within and on $|z| = 1$.

$$f(z)=\frac{\sin \pi z+\cos \pi z}{(z-1)}$$

Then *f* is analytic in the domain bounded by C_1 and C_2 and z = 2 lies in that domain . So, by Cauchy integral formula for multiply connected domain.



$$f(2)=\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{z-2}dz-\frac{1}{2\pi i}\int_{C_1}\frac{f(z)dz}{z-2} \text{ .... (i)}$$

$$=\frac{\sin 2\pi+\cos 2\pi}{2-1}=\frac{1}{2\pi i}\int_{|z|=3}\frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)}-\frac{1}{2\pi i}\int_{|z|=3/2}\frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)}$$

Again let $g(z)=\dfrac{\sin \pi+\cos \pi}{z-2}$

Then g is analytic within and on $|z|=3/2$. So, by Cauchy integral formula

$$f(1)=\frac{1}{2\pi i}\int_{|z|=3/2}\frac{g(z)}{z-1}$$

$$\Rightarrow \frac{\sin \pi z+\cos \pi z}{1-2}=\frac{1}{2\pi i}\int_{|z|=3/2}\frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)}dz$$

$$\Rightarrow 1=\frac{1}{2\pi i}\int_{|z|=3/2}\frac{\sin \pi z+\cos \pi z}{(z-1)(z-2)}dz$$

Proof : Since f is continuous in D and
for every closed contour C in D . we have by theorem.1 of section 4.4.1 that f has antiderivative F in D. But antiderivative is an analytic function, and we have just concluded that derivative of an analytic function is analytic. Consequently f is analytic in D.

1.5.3 **Liouville's theorem**

Here we will show that no entire function except a constant is bounded in the complex plane. First we prove the following simple result known as Cauchy's inequality.

Result (Cauchy's inequality) : If f is analytic within and on a circle C centred at a. and radius R, such that $|f(z)| \leq M$, $\forall z \in C$, then

$$\left|f^{(n)}(a)\right| \leq \frac{n!M}{R^n} \qquad (n = 1, 2, ...)$$

We have

$$f^{(n)}(a) \leq \frac{n!}{2\pi i}\int_C \frac{f(z)d(z)}{(z-a)^{n+1}} \qquad (n = 1, 2, ...)$$

which gives

$$f^{(n)}(a) \leq \frac{n!}{2\pi i}\int_C \frac{|f(z)|}{|z-a|^{n+1}}d(z)$$

$$\Rightarrow \left|f^{(n)}(a)\right| \leq \frac{n!}{2\pi i}\frac{M}{R^{n+1}}2\pi R$$

$$\Rightarrow \left|f^{(n)}(a)\right| \leq \frac{n!M}{R^n}$$

Theorem 1 (Liouville's theorem) **:** If f is entire and bounded in C then $f(z)$ is constant in ***C***.

that is h is an entire bounded function. Hence by Liouvilli's theorem $h(z)$=constant

$$h(z) = c \text{ (constant)}$$

$$\Rightarrow \frac{f(z)}{g(z)} = c$$

$$\Rightarrow f(z) = cg(z)$$

1.5.4 **Fundamental theorem of Algebra**

Statement : A polynomial (of positive degree) over C has at least one zero in ***C*** That is if

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots\dots\dots + a_n z^n, \ a \neq 0, \ n \geq 1$$

then there exists a point $a \in \mathbf{C} : P(a) = 0$

Proof : If possible $P(z) \neq 0 \quad \forall z \in C$,
then the function

$$Q(z) = \frac{1}{P(z)} \text{ is an entire function.}$$

Now $P(z)$ beeing a polynomial is non constant and entire. Hence by Corallary 1 of the previous section $P(z)$ is unbounded.

So,

$$\underset{z \to \infty}{Lt} P(z) = \infty \Rightarrow \underset{z \to \infty}{Lt} Q(z) = 0$$

So, we can find an $\mathbf{R} > 0 : |Q(z)| \leq 1, \qquad \forall z : |z| > \mathbf{R}$

Also Q (z) being continuous, is bounded in $|z| \leq R$ Thus $Q(z)$ is an entire bounded function, by Liouvilli's theorem. That means $P(z)$ is constant. But this is a contradiction. Hence there exists some $a \in C : P(a) = 0$

Exercise : A polynomial of degree $n(n \geq 1)$ has n zeros.

$$= f(a) = \frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i\theta}\right) d\theta$$

Hence the theorem

***Theorem* 2 :** If *f* is analytic in the neighbourhood $|z-a| < \in$ such that $|f(z)| \le |f(a)|$, $\forall z$ Then *f*(*z*) has constant value *f*(*a*) throughout that neighbourhood.

Proor : Let $c : |z-a| = r$, $r < \in$

then by the previous theorem

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{it}\right) dt$$

$$\Rightarrow |f(a)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(a + re^{it}\right)\right| dt$$

$$\Rightarrow |f(a)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(a + re^{it}\right)\right| dt \le \frac{1}{2\pi}\int_0^{2\pi} |f(a)| dt$$

$$\left[\because |f(z)| \le |f(a)| \forall z\right]$$

$$\Rightarrow |f(a)| \le \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(a + re^{it}\right)\right| dt \le |f(a)|$$

$$\Rightarrow |f(a)| = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(a + re^{it}\right)\right| dt$$

$$\Rightarrow \int_0^{2\pi} \left[|f(a)| - \left|f\left(a + re^{it}\right)\right|\right] dt = 0$$

The integral being non-negative continuous function.

we conclude $\left|f\left(a + re^{it}\right)\right| = |f(a)|$

Thus $|f(z)| = |f(0)|$ $\forall z : |z-a| = r$

But *r* is any number : $0 < r < \in$

Then $|f(z_1)|$ is the maximum in N_1. So by previous theorem f is constant in N_1, i.e. $f(z) = f(z_1), \ \forall z \in N_1$

But $z_2 \in N$, $\therefore f(z) = f(z_1) = f(z_2)$

Proceding in this way, we find $f(z_0) = f(z_n)$

i.e. $f(a) = f(b)$

But b is an arbitrary point in D.

Hence $f(z) = f(a) \ \ \forall z \in D$

Consiquently f is constant in D.

Further, suppose f is continuous on the boundary of D. Then f is continuous on $\bar{D}$. Consider the function $|f|$. Now $|f|$ is continuous on $\bar{D}$ which is closed and bounded. We know that a continuous real-valued function defined on a closed and bounded subset attains its maximum. Since $|f|$ does not attain its maximum in D, so it must attain its maximum on the boundary.

Example 1 : Let $u(x, y)$ be a non-constant harmonic function in a domain D, such that it is continuous on ∂D. Then $|u(x,y)|$ attains its maximum in ∂D.

Let $v(x,y)$ be the harmonic conjugatie of $u(x, y)$ in D. Then $f=u+iv$ is analytic in D and continuous in ∂D. Let $g(z) = e^{f(z)}$. Then $g(z)$ is non-constant and analytic in D and continuous on ∂D. Hence maximum of $|g(z)|$ occurs on ∂D. But $|g(z)| = \left|e^{f(z)}\right| = e^{u(x,y)}$ and $e^{u(x,y)}$ is an increasing function of $u(x,y)$. Hence maximum of $|u(x,y)|$ occurs on ∂D.

Example 2 : Let $f(z) = u(x,y) + iv(x,y)$ be analytic in a domain D and continuous on ∂D. Then the function $(x^2 + y^2)e^{u(x,y)}$ will have maximum on ∂D.

D. If $f(z)=0$ for same $z \in \partial D$, then minimum of $|f(z)|$ is zero and it is attained on ∂D. Otherwise $g(z)=\frac{1}{f(z)}$ is non-constant analytic in D and continuous on ∂D. Hence maximum of $|g(z)|$ is $\left|\frac{1}{f(z)}\right|$ is attained on ∂D. So minimum of $|f(z)|$ occurs on ∂D.

Example 1 : If $f(z)=0$ at some point in D, then $|f(z)|$ need not attain its minimum on ∂D. Let $f(z)=z$ and $D=\{z|z|<1\}$. Then $f(0)=0$ on

On $\partial D=\{z:|z|=1\}$, $z=e^{i\theta}$, $(0 \le \theta \le 2\pi)$

So on ∂D, $f(z)=z=e^{i\theta}$

$\Rightarrow |f(z)| \neq 0$ on ∂D

So, minimum of $|f(z)|$ does not occur on ∂D.

Example 2 : Let $|f(z)|$ be non constant analytic in a domain $D=\{z|z|<r\}$ and continuous on ∂D such that $|f(z)|>m$ on ∂D. If $|f(0)|<m$ show that there exists atleast one zero of $f(z)$ in D.

Solution : Suppose $f(z) \neq 0$ $\forall z \in D$. Then by minimum modulus theorem, minimum of $|f(z)|$ occurs on $\partial D=\{z:|z|=r\}$. But this is a contradiction, since we have $|f(0)|<m$ and $|f(z)|>m$ on ∂D. Hence $f(0)=0$ for at least on z in D.

Check your progress :

(h) Let $f(z)$ be analytic in a domain D and continuous on ∂D. If $|f(z)|=k$, $\forall z \in \partial D$ show that $f(z)$ vanishes at least once in D.

If equality hold in any of the above inequalities, then $|g(z)|$ attains maximum inside D. Then we get $g(z)$ = constant = k (say) which gives $f(z) = kz$.

Exercise : Let $f(z)$ be analytic in $|z-a|<R$ having a zero at 'a' and suppose $|f(z)|\le M \ \forall z:|z-a|<R$. Then

$$|f(z)|\le \frac{M|z-a|}{R}, \qquad \forall z:|z-a|<R$$

and $|f'(a)|\le \frac{M}{R}$

Solution : Consider the function $g(z)=f(z+a)$ then g is analytic in $|z|<R$ such that $g(0)=0$ Also $\forall z:|z|<R$, $|g(z)|=|f(z+a)|\le R$, $\because |(z+a)-a|<R$. So, we can apply Schwarz Lemma to $g(z)$ then

$$|g(z)|\le \frac{M|z|}{R}, \ \forall z:|z|<R$$

and $|g'(a)|\le \frac{M}{R}$

$$\Rightarrow \left|f(z+a)\le \frac{M|z|}{R}\right|, \ \forall z:|z|<R$$

and $|f(z+0)|\le \frac{M}{R}$

$$\Rightarrow |f(z)|\le \frac{M(z-a)}{R}, \ \forall z:|z-a|<R$$

and $|f'(z)|\le \frac{M}{R}$, replacing z by (z - a).

1.8 Let us Sum up

In this section we have seen how the concept of integration of real function can be extended to complex

3. References :

(i) R.V. Churdvill & J.w. Brewn, complex Variables and Applications, Mc. Grow Hill.

(ii) Murray R. Spiegel, theory and Problems of Complex Variables (schaum's outline Service). Sl (Metric) edition, 1981, Mc. Grow Hill.

(iii) H.S. Kasana, Complex variables, Theory and Applications, Prentice Hall of Inda.

4. Possible Answers to the CYP :

(a) Here $z(t) = t + it^2, \quad 0 \le t \le 1$

$$\therefore z'(t) = (1 + 2it)\,dt$$

$f(z) = \bar{z}, \ \therefore f(z(t)) = \overline{z(t)} = t - it^2$

so, $\int_C \bar{z}\,dz = \int_0^l (t - it^2)(1 + 2it)\,dt$ (using definition of integral)

(b) **(i)** $f(z) = e^{\pi z}$ has antiderivative $\dfrac{e^{\pi z}}{\pi}$. Use theorem 1 of section 4.4.1.

(ii) $f(z) = (z-2)^3$ has antiderivatives $\dfrac{(z-2)^4}{4}$. Use theorem 1 of section 4.4.1.

(c) $f(z) = z^2 + \sin z$ is analytic everywhere and $f'(z) = 2z + \cos z$ is continuous everywhere.

5. Model question :

1. Evaluate the following integrals :

(a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$, (b) $\int_0^{\pi/6} e^{2it} dt$,

(c) $\int_0^{\infty} e^{-3t} dt \left(\operatorname{Re} z > 0\right)$

2. Let $w(t) = u(t) + iv(t)$ denote a continuous complex valued function defined on $[-a, a]$.

(a) If $w(t)$ is even, show that

$$\int_{-a}^{a} w(t)\,dt = 2\int_0^a w(t)\,dt$$

(b) If $w(t)$ is odd, show that

$$\int_{-a}^{a} w(t)\,dt = 0$$

3. Evaluate

(a) $\int_c \frac{z+2}{z} dz$ where C is the semi circle given by

$z = 2e^{i\theta}$, $\left(0 \le \theta \le \pi\right)$

(b) $\int_c (z-1) dz$ where C is the arc from $z=0$ to $z=2$

consisting of the semi-circle $z = 1 + e^{i\theta}$, $\left(\pi \le \theta \le 2\pi\right)$

4. Evaluate

$$\int_c f(z) dz$$

where $f(z) = \begin{cases} 1, & y < 0 \\ 4y, & y > 0 \end{cases}$

and C is the arc from $z=-1-i$ to $z=1+i$, along the curve $y = x^3$.

$|f(o)| > m$ and $|f(z)| \le m \ \forall z \in \partial R$. If $f(z)$ is analytic in R, show that there exists at least one point on ∂R where the function $f(z)$ is not continuous.

11. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D and continuous on ∂D. Then show that the function : $(u^2 + v^2)e^{u(x,y)}$ will have maximum on the boundary of the domain.

12. Suppose $u(x, y)$ is harmonic function in a domain D and continuous on ∂D. Then show that $u(x, y)$ attains its maximum on the boundary of D. Furthere if $u(x,y)$ does not vanish in D, show that the minimum also occurs on the boundary of D.

13. Let $u(x, y) = 4xy + x + 1$ be defined in $|z| \le 2$. Show that $|u(x, y)|$ assume its maximum on $|z| = 2$, while the minimum occurs in $|z| < 2$. Explain.

14. Let $f(z)$ and $g(z)$ be analytic in a domain D and continuous on ∂D. If $f(z)$ and $g(z)$ coincide on the boudary of D, then show that $f(z)$ and $g(z)$ also coincides in D.

15. Let f be analytic in $|z| < 5$, and suppose that $|f(z)| \le 10$ for all points on the circle $|z-1| = 3$. Find a bound for $|f^{(3)}(0)|$.

16. Let $f(z)$ be an entire function with

$$f(z) = f(z+1) = f(z+i) \text{ for each z.}$$

show that $f(z)$ is constant.

BLOCK - 2 : SERIES [12 Marks]

STRUCTURE

2.0 **Objective**

- ❖ After going through this block will be able to
- ❖ Obtain Taylor expansion of certain functions
- ❖ Obtain Laurent expansion of certain functions in deleted neighbourhood.
- ❖ Identify singularities of a function.
- ❖ Obtain residue of a function at its singularities.
- ❖ Obtain Contour Integration Using Cauchy's Residue theorem

- Evaluate some special kinds of real integrals
- Prove Rouche's theorem applying Cauchy's Residue theorem.
- Prove Argument Principle applying Rouche's theorem.

2.1 Introduction

In Block 2 you have studied about power series and its uniqueness. In this unit you will see that if a function is analytic at a point then it has a power series representation in some neighbourhood of the point. Even for a function analytic in a deleted neighbourhood we will obtain series representation. The first type will be called Taylor's series and the second type will be called Laurent's series. You know about singularities. Here we will discuss different types of isolated singularities. We will define residue of function at its singularities. We will develop methods to find residue. The most Important result of this unit is the Cauchy's residue theorem. We will use this theorem to evaluate certain real integrals. We will all discuss Argument principle and Rouche's theorem as consequence to Cauchy's residue theorem.

2.2 Taylor Series

We already observe that a power series interior to its circle of convergence represents a differntiable function. If the power series is complex. it will represent an analytic function. In this section we will obtain the converse of that result.

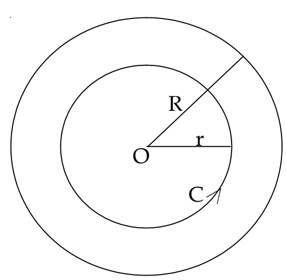
***Theorem 1* :** If f is analytic at a point z, then there exists a neighbourhood $|z-z_1|<R$ of z_0 where $f(z)$ has power series representation

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n,\quad |z-z_1|<R \text{ .... (i)}$$

such that $a_n=\dfrac{f^n(0)}{n!}$, $n=1,2$, n = 1, 2,

Proof : Since f is analytic at z_0 then exists a neigbourhood $|z-z_0|<R$ of z_0, where f is analytic. We will prove the result for z_0 =0. Then the neighbourhood is $|z|$<R. Let C denote the positively oriented circle $|z|=r<R$.

i.e $\quad z=re^{it},\qquad 0\le t\le 2\pi$



Then by Cauchy's integral formula

$$f(z)=\frac{1}{2\pi i}\int_c\frac{f(s)}{s-z}ds,\quad \forall z:|z|<r$$

$$=\frac{1}{2\pi i}\int_c\frac{f(s)\,ds}{s\left(1-z/s\right)}$$

$$= \frac{1}{2\pi i}\int_c \frac{f(s)}{s}\left(1 - z/s\right)^{-1} ds$$

$$= \frac{1}{2\pi i}\int_C \frac{f(s)}{s}\sum_{n=0}^{\infty}\left(z/s\right)^{n} ds\ ,\ \forall z : |z| < r$$

now $\left|z/s\right| = \frac{|z|}{|s|} = \frac{|z|}{r} < r$

So, we can find $M : 0 < M < 1$ and $\left|z/s\right| < M$

We have $\left|z/s\right|^{n} < M^{n} \quad (0 < M < 1)$

and $\sum_{0}^{\infty} M^{n}$ converges.

Hence by Werierstress M-test the series $\sum_{n=0}^{\infty}\left(z/s\right)^{n}$ converges uniformly in $|z| \le r$

So using theorem- 2 of section 3.4, Unit III
we have

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty} z^{n}\int_c \frac{f(s)\,ds}{s^{n+1}}, \quad \forall z : |z| < r$$

$$= \sum_{n=0}^{\infty} z^{n}\,\frac{1}{2\pi i}\int_c \frac{f(s)\,ds}{s^{n+1}}$$

$$= \sum_{n=0}^{\infty} z^{n}\,\frac{f^{n}(0)}{n!} = \sum_{n=0}^{\infty}\frac{f^{n}(0)}{n!} z^{n}$$

Since C can be any circle with radius *r*<*R*, the result obtained is valid $\forall z : |z| < R$. If *f* is an antire function, than $R = \propto$.

Representation (i) is called Tayler series of *f*(*z*) about z_0. For $z_0 = 0$, it is called Maclaurin series for *f*(*z*) .

Example 1 : Consider the Maclarin series of the function :

(a) e^{z}, (b) $sin\, z$, (c) $\frac{1}{1-z}$

Solution : (a) $f(z) = e^z$ is an entire function, so the series expansion will be valid for all z in the complex plane.

As $\quad f^n(z) = e^z \quad \forall n$

$= f^n(0) = 1, \quad \forall n$

Hence Maclaurin, series is $\sum_{n=0}^{\alpha} \frac{z^n}{ni}$

(b) $f(z)$= sin$z$ is also an entire function.

Here $\quad f^I(z) = \cos z, \quad f^{II}(z) = -\sin z, \quad f^{III}(z) = -\cos z,$

$f^{IV}(z) = \sin z$

So, $f^{(2n+1)}(0) = (-1)^n$ and $f^{2n}(0) = 0$, n = 0, 1, 2, ...

Hence Maclaurin series is $\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$

(c) $f(z) = \frac{1}{1-z} = (1-z)^{-1}$

$= \sum_{n=0}^{\infty} z^n, \qquad |z| < 1$ (using binomial expansion)

Example 2 : Find Taylor expansion of cosz about $z = \pi/2$

Solution : We have $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \qquad \forall z : |z| < \infty$

Replacing z by $\quad z - \pi/2$, we get

$$\sin\left(z - \pi/2\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(z - \pi/2\right)^{2n+1}}{(2n+1)!}, \quad \forall z : |z| < \infty$$

i.e. $$-\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n \left(z - \pi/2\right)^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \cos z = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \left(z - \pi/2\right)^{2n+1}}{(2n+1)!}, \quad \forall z : |z| < \infty$$

Note : You can also get the above expansion by finding $f^{n}\left(\pi/2\right)$ (*n = 0, 1, 2, ...*) for the function $cos\,z$.

> **C heck your rogress:**
>
> (a) Find the Maclaurin series for $cos\,z$
>
> (b) Find the Taylar series for $sin\,z$ about $z=-\pi$

2.2.1 **Zeros of an analytic function :**

Definition **:** Let f be analytic at a point z_0, suppose $f(z_0)=f^{I}(z_0)=f^{II}(z_0)=.....=f^{(m-1)}(z_0)=0$ but $f^{(m)}(z_0)\neq 0$, but $f^{(m)}(z_0)\neq 0$ then f is said to have a zero of order m at z_0.

Theorem 1 **:** Let *f* be analytic at z_0.

Then $z_0 \Leftrightarrow f(z)=(z-z_0)^{m}g(z).....(i)$

where g is analytic at z_0 and $g(z_0)\neq 0$.

Proof : Since *f* is analytic at z_0, there exists a neighbourhood $|z-z_0|<R$ of z_0 where *f*(*z*) has Taylor series expansion.

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{n}(z_0)}{n!}(z-z_0)^{n},\quad \forall z:|z-z_0|<R$$

Suppose *f* has a zero of order *m* at z_0, then $f^{n}(z_0)=0$ for $n = 0, 1, 2,....(m+1)$

$$\therefore f(z)=\sum_{n=m}^{\infty}\frac{f^{n}(z_0)}{n!}(z-z_0)^{n},\qquad \forall z:|z-z_0|<R$$

$$=\frac{f^{m}(z_0)}{m!}(z-z_0)^{m}+\frac{f^{m+1}(z_0)}{(m+1)!}(z-z_0)^{m+1}+..........$$

$$=(z-z_0)^{m}\left[\frac{f^{m}(z_0)}{m!}+\frac{f^{m+1}(z_0)}{(m+1)!}(z-z_0)+..........\right]$$

$$=(z-z_0)^{m}.g(z)$$

where $g(z) = \frac{f^{m}(z_0)}{m!} + \frac{f^{m+1}(z_0)}{(m+1)!}(z - z_0) + \ldots.$

is analytic at z_0 and $g(z_0) = \frac{f^{m}(z_0)}{m!} \neq 0$

Conversely, Let *f*(z) has expression (i). As *g*(*z*) is analytic at z_0, there exists some neighbourhood $|z - z_0| < R$ of z_0 , where *g*(*z*) has Taylor series expansion

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{n}(z_0)}{n!}(z - z_0)^{n}, \quad \forall z : |z - z_0| < R$$

Then by (i)

$$f(z) = (z - z_0)^{m} \sum_{n=0}^{\infty} \frac{g^{n}(z_0)}{n!}(z - z_0)^{n}, \ \forall z : |z - z_0| < R$$

By uniqueness of Power series, the above is the Taylor series for *f*(*z*). Hence we have

$$f(z_0) = f^{i}(z_0) = f^{ii}(z_0) = \ldots. = f^{(m-1)}(z_0) = 0$$

and $f^{(m)}(z_0) = m!\,g(z_0) \neq 0$

consequently *f* has a zero of order m at z_0.

An important conclusion of the above theorem is the following Corollary.

Corollary : Zeros of an analytic function are isolated.

Proof : Let z_0 be a zero of order *m* of an analytic function *f*. Then there exists some neighbourhood $|z - z_0| < R$ of z_0 such that

$$f(z) = (z - z_0)^{m} g(z)$$

where *g*(*z*) is analytic at z_0 and $g(z_0) \neq 0$. As *g* is analytic at z_0, it is continuous at z_0. As $g(z_0) \neq 0$, there exists a neighbourhood of z_0, say $|z - z_0| < R$ where $g(z) \neq 0$.

Consequently

$$f(z) = (z - z_0)^m g(z) \neq 0, \ \forall z : 0 < |z - z_0| < r$$

Thus, $f(z) \neq 0$ in a deleted neighbourhood of z_0, i.e. z_0 is an isolated singularity of *f*.

Corollary : If *f*(*z*) and *g*(*z*) are analytic at z_0 and have zeroes of order *m* and *n* respectively at *z*= z_0, then $h(z) = f(z)g(z)$ has a zero of order *m* + *n* at z_0.

Check your progress

(c) Find order of zeroes of the function $z^3 \sin z$ at the origin.

(d) Locate the zeroes of the function and determine their order

(a) $(1 + z^2)$, (b) $z^3 + z$

2.3 **Laurent Series**

If *a* function fails to be analytic at a point *a*, then we cannot apply Taylors theorem. However it is possible to find a series representation of *f* called Laurent Series expansion involving both positive and negative powers of (*z* - *a*).

Theorem : Let *f* be analytic throughout an annular domain $R_1 < |z - a| < R_2$ centred at *a*, and C denote any positively oriented simple closed contour around *a* and lying in that domain. Then at each point in the domain *f*(*z*) has the series representation.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - a)^n}, \quad \forall z : R_1 < |z - a| < R_2 \ldots \text{(i)}$$

where

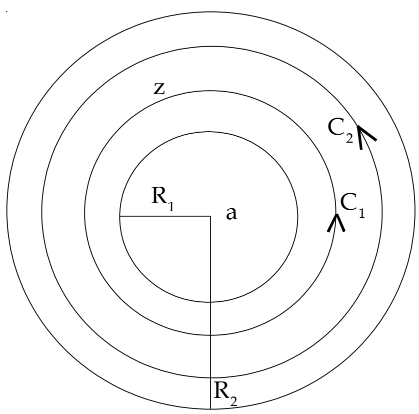
$$a_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(s-a)^{n+1}}, \quad n = 0, 1, 2, \ldots\ldots\ldots\ldots$$

$$b_n = \frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{(s-a)^{-n+1}}, \quad n = 1, 2, \ldots\ldots\ldots\ldots\ldots$$

Series (i) is called Laurent series expansion of $f(z)$ about $z = a$.

$\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the regular part and $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ is called the singular part of $f(z)$.

Proof : Consider the circle $c_1 : |z-a| = r_1$ and $c_2 : |z-a| = r_2$ such that $R_1 < r_1 \le |z-a| \le r_2 < R_2$.



By Cauchy integral formula for doubly connected domain, we have

$$f(z) = \frac{1}{2\pi i}\int_{c_2} \frac{f(s)\,ds}{s-z} - \frac{1}{2\pi i}\int_{c_1} \frac{f(s)\,ds}{s-z}$$

Consider the first integral,
we have

$$\frac{1}{s-z} = \frac{1}{(s-a)\left(1 - \frac{z-a}{s-a}\right)}$$

$$= \frac{1}{(s-a)}\left(1-\frac{z-a}{s-a}\right)^{-1}$$

$$= \frac{1}{(s-a)}\left(1+\frac{z-a}{s-a}+\left(\frac{z-a}{s-a}\right)^{2}+.........\right)$$

$$= \frac{1}{(s-a)}+\frac{z-a}{(s-a)^{2}}+\frac{(z-a)^{2}}{(s-a)^{3}}+..........$$

So,

$$\frac{1}{2\pi i}\int_{c_2}\frac{f(s)ds}{s-z}=\frac{1}{2\pi i}\int_{c_2}\frac{ds}{s-a}+\frac{z-a}{2\pi i}\int_{c_2}\frac{ds}{(s-a)^{2}}+\frac{(z-a)^{2}}{2\pi i}\int_{c_2}\frac{ds}{(s-a)^{3}}+...$$

$$= a_0 + a_1(z-a)+a_2(z-a)^2+...$$

where

$$a_n = \frac{1}{2\pi i}\int_{c_2}\frac{ds}{(s-z)^{n}}$$

Next consider the scond integral.
We have

$$\frac{1}{s-z}=\frac{1}{-(z-a)\left(1-\frac{s-a}{z-a}\right)}$$

$$= -\frac{1}{(z-a)}\left(1-\frac{s-a}{z-a}\right)^{-1}$$

$$= -\frac{1}{(z-a)}\sum_{n=0}^{\infty}\left(\frac{s-a}{z-a}\right)^{n}$$

So,

$$\frac{1}{2\pi i}\int_{c_1}\frac{f(s)ds}{s-z}=-\frac{1}{2\pi i}\int_{c_1}\frac{f(s)}{z-a}\sum_{n=0}^{\infty}\left(\frac{s-a}{z-a}\right)^{n}ds$$

$$= -\frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_{c_1}\frac{f(s)(s-a)^{n}}{(z-a)^{n+1}}ds,\quad \left[\sin ce\left|\frac{s-a}{z-a}\right|<1\right]$$

$$= -\sum_{n=0}^{\infty} \frac{1}{(z-a)^{n+1}} \frac{1}{2\pi i} \int_{c_1} f(s)(s-a)^n ds$$

$$= -\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $b_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(s)\,ds}{(s-a)^{-n+1}}$

By the principle of deformation of path c_1 and c_2 may be replaced by a simple closed contour c entirely lying in the annular domain.

Remark : If f fails to be analytic at a , but is otherwise analytic in the disk $|z-a| < R_2$, the radius R_1 can be chosen arbitarily small. Then validity of the series is $0 < |z-a| < R_2$. Again if f is analytic at each point in the finite plane exterior to the circle $|z-a| = R_1$, then the condition of validity is $R_1 < |z-a| < \propto$. Also if f is analytic everywhere in the finite plane except at a, then the series is valid in the domain $0 < |z-a| < \propto$.

Example : Obtain the Laurent series of the functions :

(a) $e^{1/z}$, (b) $z^3 \cosh 1/z$, (c) $\frac{1}{z^2(1-z)}$

Solution : (a) we have Maclaurin series for e^z as

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{ni}, \qquad |z| < \propto$$

Replacing z by $1/z$. we have Laurent series for $e^{1/z}$ as

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{z^n ni}, \; 0 < |z| < \propto$$

(b) We have replacing z by $1/z$ in Maclaurin series for *cos*hz

$$\cosh 1/z = \sum_{n=0}^{\infty} \frac{1}{z^{2n}\, 2n!}, \qquad 0<|z|<\infty$$

$$\therefore z^3 \cosh 1/z = \sum_{n=0}^{\infty} \frac{z^{3-2n}}{(2n)!}, \qquad 0<|z|<\infty$$

$$= z^3 + z/2 + \sum_{n=0}^{\infty} \frac{1}{(2n)!\, z^{2n-3}} \qquad 0<|z|<\infty$$

(c) The function is not analytic at *z=0* and *z=1*.

First consider the domain $0<|z|<1$.
then

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2}(1-z)^{-1} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n$$

$$= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^{n-2}, \quad 0<|z|<1$$

Next consider the domain $1<|z|<\infty$.
Then

$$\frac{1}{z^2(1-z)} = -\frac{1}{z^3\left(1-1/z\right)}$$

$$= -\frac{1}{z^3}\left(1-1/z\right)^{-1}$$

$$= \frac{1}{z^3}\sum_{n=0}^{\infty}\left(1/z\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}}, \quad 1<|z|<\infty$$

Check your progress :

(e) Derive Laurent series for the functions

(i) $\dfrac{z-\sin z}{z^4}$, (ii) $z^2 \sin 1/z^2$

2.3.1 **Classification of singularities :**

You know that any singularity of a function can be classified into either isolated singularity or non-isolated singularity. In our study we will be mainly interested in isolated singularities. There are three types of isolated singularities- removable singularity, pole and essential singularity.

Suppose a is an isolated singularity of f then there exists a deleted neighbourhood of a, $|z-a|<R$ where $f(z)$ has Laurent series represefntation.

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n},\quad 0<|z-a|<R \quad \text{...... (i)}$$

Definition **(Removable singularity)** : If the principal part of (i) is zero, z=a is said to be removable singularity.

e.g. $f(\mathrm{z})=\dfrac{\sin \mathrm{z}}{\mathrm{z}}$

Pole : If the principal part has finite number of terms, then z = a is a pole of f.

eg. $f(z)=\dfrac{1}{(z-2)^2}$. Then z=2 is a pale of f.

If m is the largest integer such that $b_m \neq 0$, then f is said to have a pole of order m. In the above example z=2 is a pole of order 2. A pole of order 1 is called a simple pole. For the function.

$$f(z)=\frac{z^2-2z+3}{z-2}=z+\frac{3}{z-2},\qquad 0<|z-2|<\infty$$

z = 2 is a simple pole.

Definition **(Essential singularity) :** If the principle part contains infinite number of terms, then z=a is called an essential singularity.

Eg. $f(z)=e^{1/z}=1+\frac{1}{z}+\frac{1}{2!z^2}+\frac{1}{3!z^3}+\ldots,\qquad 0<|z|<\infty$

So, it has $z=0$ as essential singularity.

2.4 Residues

Definition **:** Let a be an isolated singularity of a function *f*. Then there exists a deleted neighbouhood $0<|z-a|<R$ where $f(z)$ has Laurent series expansion.

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n},\ 0<|z-a|<R$$

Here *b*, i.e. the coefficient of $\frac{1}{z-a}$ is called the residue of f at its singularity *a*.

We know the coefficient b_n's are given by

$$b_n=\frac{1}{2\pi i}\int_c\frac{f(z)dz}{(z-a)^{-n+1}},\ n = 1, 2, 3, \ldots\ldots\ldots\ldots\ldots$$

Where *c* is any single closed contour lying in the annulus $0<|z|<R$.

Hence residue at $z=a$.

$$b_1=\frac{1}{2\pi i}\int_c f(z)dz \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Thus we observe that if the contour integral is known, the residue can be found and conversely.

Example : The function $f(z)=e^{1/z^2}$ has singularity only at z = 0. By Laurent series expansion.

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{z^2n!},\qquad 0<|z|<\infty$$

Residue at z = 0, i.e. coefficient $\frac{1}{z} = 0$

Consequently $\int_c f(z)\,dz = 2\pi i.0$ for any

simple closed contour *c* encircling the origin.

Theorem 1 **:** If a function *f*(*z*) has an isolated singularity at *a* and *f*(*z*) is even in (*z* - *a*), i.e. *f*(*z* - *a*) = *f*(-(*z* - *a*)),

then

$$\operatorname*{Res}_{z=a} f(z) = 0$$

Proof : Since *f* is even, the Laurent series expansion of *f* about a cannot have odd power of (*z* - *a*). Hence the result

Example : Consider the function $f(z) = \frac{\sin z^2}{z^2 \sin z}$

It has isolated singularity at *z=0*

As *f* is even $\operatorname*{Res}_{z=o} f(z) = 0$

2.4.1 **Cauchy residue theorem :**

We know from Cauchy's theorem that if a function f is analytic within and on a simple closed contour C, then integral of *f*(*z*) around the contour is zero. However if *f* has finite number of singularities within C, then the value of the integral depends on the residue of *f* at its singularities within C. The following theorem pricisely states that.

Theorem 1 **(Cauchy residue theorem)** : Let C be a simple closed contour described in positive sense. If *f* is analytic within and on C except for finit number of singularities within C, then

$$\int_c f(z)\,dz = 2\pi i \text{ [sum of the residue of } f \text{ at the singularities]}$$

Proof : Let the singularities of f within C be $z_1, z_2, \ldots. z_k$. We consider positively oriented circles C_r centred at each z_r which are interior to C and no two of them have points in common. Then the circles C_1, together with the simple closed contour C, form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain, Then by Cauchy-Goursat theorem for multiply connected region we get

$$\int_c f(z)\,dz - \sum_{r=1}^{k} \int_{c_r} f(z)\,dz = 0$$

$$\Rightarrow \int_c f(z)\,dz = \sum_{r=1}^{k} \int_{c_r} f(z)\,dz$$

$$= 2\pi i \sum_{r=1}^{k} \operatorname*{Res}_{z=z_R} f(z) \text{ using (i)}$$

Thus from the above theorem we observe that we can find the value of an integral without practically evaluating it. You need simply to locate the singularities of the function within the contour and find the residues at these singularities.

Example : Consider the function $f(z) = \frac{1}{z-1}$

The only singularity of $f(z)$ is at $z=1$.

$f(z)$ is in Laurent series form, so the residue at $z=1$ is observed to be 1.

Hence $\int_c f(z)\,dz = 2\pi i$ for any simple closed contour C around the point $z=1$.

If f is analytic within, on and outside a simple closed contour C, except a finite number of singularities within C, then the integral of f can be evaluated around c simple by finding the residue of a certain related function at the origin. The fact is proved in the following theorem.

Theorem 2 : If f is analytic everywhere in the finite complex plane except for a finit number of singularities within a simple closed contour C, then

$$\int_c f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left\{\frac{1}{z^2} f\left(1/z\right)\right\}$$

Proof : Consider the circle $|z| = R_1$ such that the contourf c lies interior to it. Let $c_0 = |z| = R_0$, such that $R_0 > R_1$

Then $f(z)$ has Laurent series expansion

$$f(z) = \sum_{-\infty}^{\infty} c_n z^n \ , \ R_1 < |z| < \infty \ \text{..............} \ \text{(i)}$$

where $c_n = \dfrac{1}{2\pi i}\displaystyle\int_{c_0} \dfrac{f(z)\,dz}{z^{n+1}}\ ,\ n = 0, \pm 1, \pm 2, \ldots\ldots$ (ii)

For, n = -1, we have

$$\int_{c_n} f(z)\,dz = 2\pi i c_{-1} \ \text{......................} \ \text{(iii)}$$

Replace z by $1/z$ in (ii). Then we have

$$f\left(1/z\right) = \sum_{-\infty}^{\infty} \frac{c_n}{z^n}\ , \qquad R_1 < \frac{1}{|z|} < \infty$$

$$\Rightarrow \frac{1}{z^2} f(z) = \sum_{-\infty}^{\infty} \frac{c_n}{z^{n+2}}\ ,\ \ 0 < |z| < 1/R_1$$

This is the Laurent series expansion of $\dfrac{1}{z^2} f(z)$ around its singularity *z=0*

$$\therefore \operatorname*{Res}_{z=0} \frac{1}{z^2} f(z) = c_{-1} \quad \text{......................} \ \text{(iv)}$$

From (iii) and (iv)

$$\int_{C_0} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left\{\frac{1}{z^2} f\left(1/z\right)\right\}$$

Since f is analytic throughout the closed region bounded by c and c_0, by principle of deformation of paths, we get

$$\int_{C_0} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left\{\frac{1}{z^2} f\left(\frac{1}{z}\right)\right\}$$

Example : Evaluate

$$\int_c \frac{5z-2}{z(z-1)}\,dz$$

using single residue method, where $c:|z|=2$

Here $f(z)=\dfrac{5z-2}{z(z-1)}$

we have

$$\frac{1}{z^2} f\left(\frac{1}{z}\right)=\frac{\frac{5}{z}-2}{\frac{1}{z}\left(\frac{1}{z}-1\right)}$$

$$=\frac{5-2z}{z}\cdot\frac{1}{1-z}$$

$$=\left(\frac{5}{z}-2\right)(1-z)^{-1}$$

$$=\left(\frac{5}{z}-2\right)\left(1+z+z^2\ldots\ldots\ldots\ldots\right) \quad 0<|z|<1$$

$$=\frac{5}{z}+3+3z\ldots\ldots\ldots\ldots\ldots \quad 0<|z|<1$$

$$\therefore \operatorname*{Res}_{z=0}\frac{1}{z^2} f\left(\frac{1}{z}\right)=5$$

Hence

$$\int_c f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\frac{1}{z^2} f\left(\frac{1}{z}\right)$$

$$=2\pi i(5)=10\pi i$$

Check your progress

(f) Evaluate using single residue method

(i) $\int_c \frac{dz}{1+z^2}$, (ii) $\int_c \frac{z^5}{1-z^3}dz$

where $c:|z|=2$

2.4.2 **Method to calculate residue :**

In this section we will discuss some methods to evaluate residue of function at its singularities. Note that residue at a removable singularity is zero.

Method 1 : Sippose z_0 is a singularity of a function *f*. Write the appropriate Laurent series expansion of *f*(*z*) in the deleted neighbourhood of z_0. Then the coefficient of $\frac{1}{z-z_0}$ is the residue of *f*(z) at $z=z_0$

Example 1 : $f(z)=\frac{1}{1-z}$

The function has singularity at *z=1*. The *f*(*z*) is in Laurent series form, so residue of *f*(*z*) at *z*=1 is 1

Example 2 : $f(z)=\frac{cos\, z}{z}$

it has singularity at *z=0*

Writing the Maclaurin series for *cos*z we have

$$f(z)=\frac{\cos z}{z}=\frac{1}{z}\left\{1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+........\right\},\ 0<|z|<\infty$$

$$=\frac{1}{z}-\frac{z}{2!}+\frac{z^3}{4!}-\frac{z^5}{6!}+.......,\qquad 0<|z|<\infty$$

$$\therefore \operatorname*{Res}_{z=0} f(z)=1$$

Example 3 : $f(z)=\dfrac{e^{-z}}{(z-2)^4}$

z=2 is a singularity. We find the Laurent series expansion in $0<|z-2|<\propto$

$$f(z)=\frac{e^{-z}}{(z-2)^4}=\frac{e^{-2}.e^{-(z-2)}}{(z-2)^4}$$

$$=\frac{e^{-2}}{(z-2)^4}\left\{1-(z-2)+\frac{(z-2)^2}{2!}+\frac{(z-2)^3}{3!}+.......\right\}$$

$$=e^{-2}\left\{\frac{1}{(z-2)^4}-\frac{1}{(z-2)^3}+\frac{1}{2!(z-2)^2}-\frac{1}{3!(z-2)}+...\right\}$$

Hence $\operatorname*{Res}_{z=0} f(z)=-\dfrac{e^{-2}}{3!}=-\dfrac{1}{6e^2}$

> **Check your progress :**
> (g) Find the Laurent series and calculate residues
>
> (i) $\dfrac{1}{z(z+1)}$, (ii) $z\,cos\,{}^1\!/_z$, (iii) $\dfrac{sin\,hz}{z^4(1-z^2)}$

Method 2 :

Here we develop a method to calculate residue at poles. First we prove the following theorem .

Theorem : The function $f(z)$ has a pole of order m at a iff $f(z)$ can be written in the form

$$f(z)=\frac{\phi(z)}{(z-a)^m}$$

where $\phi(z)$ is analytic at a and $\varphi(a)\neq 0$

Proof : Suppose $f(z)$ has a pole of order m at a. Then there is some deleted neighbourhood $0<|z-a|<R$ where Laurent series of $f(z)$ is

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\frac{b_1}{z-a}+\frac{b_2}{(z-a)^2}+\ldots+\frac{b_m}{(z-a)^m},\quad \mathrm{b_m}\neq 0$$

which gives

$$f(z)=\frac{1}{(z-a)^m}\left\{\sum_{n=0}^{\infty}a_n(z-a)^{n+m}+b_1(z-a)^{m-1}+b_2(z-a)^{m-2}+\ldots+b_m\right\}$$

$$=\frac{\phi(\mathrm{z})}{(\mathrm{z}-\mathrm{a})^{\mathrm{m}}}$$

where

$$\phi(z)=\sum_{n=0}^{\infty}a_n(z-a)^{n+m}+b_1(z-a)^{m-1}+b_2(z-a)^{m-2}+\ldots+b_m$$

is analytic at a and $\phi(z_0)=b_m\neq 0$

Conversely, suppose $f(z)=\dfrac{\phi(z)}{(z-a)^m}$, where $\phi(a)\neq 0$ and $\phi(z)$

is analytic at a.

As $\phi(z)$ is analytic at $\phi(z)$,

$$\phi(z)=\phi(a)+\phi'(a)(z-a)+\frac{\phi''(a)}{2!}+\ldots+\frac{\phi^{(m-1)}(a)}{(m-1)!}(z-a)^{m-1}+\ldots.$$

$$\Rightarrow f(z)=\frac{\phi(a)}{(z-a)^m}+\ldots+\frac{\phi^{(m-1)}(a)}{(m-1)!(z-a)}+\frac{\phi^{m}(a)}{m!}+\frac{\phi^{(m-1)}(a)}{(m-a)!}(z-a)+\ldots.$$

$$\forall \mathrm{z}:0<|z-a|<R$$

This is the Laurent expansion of $f(z)$.

Hence $f(z)$ has a pole of order m at $z=a$.

The residue of $f(z)$ at $z=a$ is

$$\frac{\phi^{(m-1)}(a)}{(m-1)!}$$

which can be written as

$$\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]_{z=a}$$

Hence for a function $f(z)$ having pole of order m at a point $z=a$

$$\operatorname*{Res}_{z=a} f(z)=\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m=1}}\left[(z-a)^m f(z)\right]_{z=a}$$

For simple pole

$$\operatorname*{Res}_{z=a} f(z)=(z-a)^m f(a)=\phi(a)$$

Example 1 : $f(z)=\dfrac{z+1}{z^2+9}$

$f(z)$ has singularities at $z=\pm 3i$
we write

$$f(z)=\frac{z+1}{(z+3i)(z-3i)}$$

First let $\qquad \phi(z)=\dfrac{z+1}{z+3i}$

then $\qquad f(z)=\dfrac{\phi(z)}{z-3i}$

$$\operatorname*{Res}_{z=3i} f(z)=\phi(3i)=\frac{3i+1}{3i+3i}=\frac{3i+1}{6i}=\frac{3-i}{6}$$

Next let $\phi(z)=\dfrac{z+1}{z-3i}$, then $f(z)=\dfrac{\phi(z)}{z+3i}$

$$\operatorname*{Res}_{z=3i} f(z)=\phi(-3i)=\frac{-3i+1}{-3i-3i}=\frac{3i-1}{6i}=\frac{3+i}{6}$$

Example 2 : $f(z)=\dfrac{z^2+16}{(z-i)^2(z+3)}$

$f(z)$ has singularities at z = i, z = -3

First let $\phi(z)=\dfrac{z^2+16}{(z-i)^2}$

then $f(z)=\dfrac{\phi(z)}{z+3}$

$$\operatorname*{Res}_{z=-3} f(z)=\phi(-3)=\frac{9+16}{(-3-i)^2}=\frac{25}{9-1+6i}=\frac{25}{8+6i}=\frac{25(8-6i)}{64+36}=2-\tfrac{3}{2}i$$

Next let $\phi(z)=\dfrac{z^2+16}{z+3}$

Then $f(z)=\dfrac{\phi(z)}{(z-i)^2}$

$f(z)$ has pole of order 2 at $z=i$

$$\therefore \operatorname*{Res}_{z=i} f(z)=\frac{1}{(2-1)!}\frac{d}{dz}\left[(z-i)^2 f(z)\right]_{z=i}$$

$$=\frac{1}{1!}\frac{d}{dz}\left[\phi(z)\right]_{z=i}$$

$$=\left[\frac{(z+3)2z-(z^2+16)i}{(z+3)^2}\right]_{z=i}$$

$$=-1+\tfrac{3}{2}i$$

Check your progress:

(h) Find the residue at poles of the functions

(i) $\dfrac{z^2+2}{z-1}$, (ii) $\left(\dfrac{z}{2z+1}\right)^3$, (ii) $\dfrac{e^z}{z^2+\pi^2}$

Method 3 : For this method first we prove the following theorem.

***Theorem* 2:** If a function $f(z)$ is of the form $f(z)=\dfrac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are analytic at a and $p(a)\neq 0$ then $z=a$ is a pole of order m iff $q(z)$ has a zero of order m at $z=a$.

Proof : we know that $q(z)$ has a zero of order m at $z=a$ iff it is of the form

$$q(z)=(z-a)^m\phi(z)$$

where $\phi(z)$ is analytic at a and $\phi(a)\neq 0$

then

$$f(z)=\frac{p(z)}{q(z)}=\frac{p(z)}{(z-a)^m\phi(z)}=\frac{1}{(z-a)^m}\psi(z)$$

where $\psi(z)=\dfrac{p(z)}{\varphi(z)}$ is analytic at a and $\psi(a)\neq 0$.

But by the last theorem we know that the form

$$f(z)=\frac{\psi(z)}{(z-a)^m}$$

ensures that $f(z)$ has a pole of order m at $z=a$.

Let us consider the particular case of $q(z)$ having a simple zero at $z=a$. Then $q(a)=0$ but $q'(a)\neq 0$. We write $q(z)=(z-a)\phi(z)$, where $\phi(z)$ is analytic at $z=a$ and $\phi(a)\neq 0$.

Actually $\varphi(a)=q'(a)$

Then

$$f(z)=\frac{p(z)}{q(z)}=\frac{p(z)}{(z-a)\phi(z)}=\frac{\left(\dfrac{p(z)}{\varphi(z)}\right)}{(z-a)}$$

So, $f(z)$ has a simple pole at $z=a$.

By Method 2 :

$$\operatorname*{Res}_{z=a} f(z) = \frac{p(a)}{\varphi(a)} = \frac{p(a)}{q'(a)}, \quad [\because \varphi(a) = q'(a)]$$

Example 1 : $f(z) = \cot z = \dfrac{\cos z}{\sin z}$

Let $p(z) = \cos z$ and $q(z) = \sin z$

then $q(z)=0$ when $z = n\pi \ (n = 0, \pm 1, \pm 2, \ldots\ldots\ldots\ldots\ldots\ldots)$

So, $\operatorname*{Res}_{z=n\pi} f(z) = \dfrac{p(n\pi)}{q'(n\pi)} = \dfrac{(-1)^n}{(-1)^n} = 1$

Example 1 : $f(z) = \dfrac{z^3 + z + 2}{z^2 + 2z + 1}$

Let $p(z) = z^3 + z + 2$ and $q(z) = z^2 + 2z + 1$

$q(z) = 0$ when $z = 1$ and $q'(1) = 4 \neq 0$

Also $p(1) = 4 \neq 0$

Hence $f(z)$ has a simple pole at $z=1$.

So,

$$\operatorname*{Res}_{z=1} f(z) = \frac{p(1)}{q'(1)} = \frac{4}{4} = 1$$

Check your progress :

(i) Find residue at singularities of

(a) $\dfrac{z^2 + 1}{z(z+1)}$, (b) $\dfrac{\sin z}{\cos z}$, (c) $\sec z$

2.5 **Application Cauchy residue theorem**

In this section we will derive some results based on the Cauchy residue theorem. We will also use residue calculus to evaluate certain definite real integral and real improper integrals.

2.5.1 **Argument Principle**

Definition : A function whose only singularities in the finite complex plane are poles is called a meromorphic function.

All rational functions, *tan*z, *sec*z, $\frac{1}{z-1}$ are examples of meroorphic functions. By definition, every analytic function is meromorphic.

Let us consider a function *f* which is meromorphic in the interior of a simple closed curve C and such that it is analytic and has no zeros on C. Let Γ denote the image of C under *f*. As *f*(*z*) is not zero on C, Γ will not pass through the origin. As a point z traverses along the positive direction of C the image point traverses along Γ determing its orientation.

Let w_0 be an arbitrarily fixed point on Γ with $arg\, w_0 = \phi_0$ Let a point w traverse Γ once in positive direction starting from the point w_0 . Let ϕ_1 be the arg of w after it returns to the origin $\propto$ position w_0. So, the change in arg. of w as w completes one traversal along Γ in positive sense is $\phi_1 - \phi_0$.

As w= *f*(*z*) this change in arg of w is actually the change in $arg\, f(z)$ as z describes C once in the positive direction starting from any point z_0 in C. We write change is arg as $\Delta_c f(z)$

$$\therefore \Delta_c f(z) = \phi_1 - \phi_0$$

where ϕ_0 is the initial value of arg and ϕ is the value of arg after the point *z* completes one traversal along C in positive sense. Clearly this value is an integral multiple of 2π. Therefore $\frac{1}{2\pi}\Delta_c f(z)$ gives the number of times *f*(*z*) winds around the origin in the w-plane whem *z* describes C once in positive sense. Hence this number is called winding number of Γ w.r.t. the origin w=0. It is taken to be positive if Γ winds around the origin in counter clockwise sense and negative otherwise. Winding number is zero if Γ does not enclose the origin.

***Theorem 1*:** Let C be a simple closed contour. If f is memomorphic inside C, analytic and non-zero an C, then

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = N - P$$

where N = number of zeros of f within C
and P= number of poles of f within C.

Note : A zero of order m will be counted as m zeros and similarly a pole of order p will be counted as p poles. This is termed as counting multiplicity.

Proof : Suppose $f(z)$ has a zero of order m at $z=a$.
Then

$$f(z) = (z-a)^m g(z)$$

where g is analytic at a and $g(a) \neq 0$
Hence

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{g'(z)}{g(z)}$$

As $g(a) \neq 0$, $\frac{g'(z)}{g(z)}$ is analytic at z=a and hence $\frac{f'(z)}{f(z)}$ has a simple pole at z = a with residue m.

We have seen that zero of f(z) is a pole of $\frac{f'(z)}{f(z)}$ and the residue is the order of the zero. Consequently the sum of the residues at poles of $\frac{f'(z)}{f(z)}$ is N.

Next suppose $f(z)$ has a pole of order n at $z=b$, then

$$f(z) = \frac{\phi(z)}{(z-b)^n}$$

where $\phi(z)$ is analytic at $z=b$ and $\phi(b) \neq 0$

Hence

$$\frac{f'(z)}{f(z)} = -\frac{n}{z-b} + \frac{\phi'(z)}{\phi(z)}$$

As $\phi(b) \neq 0$, $\dfrac{\phi'(z)}{\phi(z)}$ is analytic at z = b and hence $\dfrac{f'(z)}{f(z)}$ has a simple pole at $z = b$ with residue - n.

As above the sum of residues at poles is -P.

Then by Cauchy residues theorem we have

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = N - P$$

Example : Consider the function $f(z) = \dfrac{(z^2+1)^2}{(z^2+3z+2)^3}$

and the simple closed contour $c : |z| = 3$.

The function has double zero at $z = i, -i$, poles of order three at $z = -1, -2$.

Hence N = 2+1 =4, P = 3+3 = 6

Poles and zeros of $f(z)$ are inside C, hence by the above theorem

$$\frac{1}{2\pi i}\int_c \frac{f'(z)dz}{f(z)} = 4 - 6 = -2,$$

Check your progress

(j) Find $\int_c \dfrac{f'(z)}{f(z)}dz$ where $C : |z| = 2$.

and $f(z) = \dfrac{z-1}{z^2+2z+1}$

The next theorem is called the Argument Principle.

***Theorem* 2 :** Let C be a simple closed contour. A function *f* is analytic and non-zero on C. meromorphic within C. If N is the number of zeros and P is the number of pole (counting multiplicaties) of $f(z)$ within C, then

$$\Delta_c \arg f(z) = 2\pi(N-P)$$

Proof : Let $z=z(t)$ $(a \le t \le b)$ be the parametric representation of C. Then.

$$\int_c \frac{f'(z)dz}{f(z)} = \int_a^b \frac{f'(z(t))z'(t)dt}{f(z(t))}$$

Let Γ be the image of C.

Since $f(z) \ne 0$ for any z on C, any point on Γ can be represented as

$$f(z(t)) = r(t)e^{i\theta(t)} \qquad (\mathrm{a} \le \mathrm{t} \le \mathrm{b})$$

Then we have

$$f'(z(t))z'(t) = \frac{d}{dz}\left[f(z(t))\right] = \frac{d}{dz}\left[r(t)e^{i\theta(t)}\right]$$

$$= r'(t)e^{i\theta(t)} + ir(t)e^{i\theta(t)}.\theta'(t)$$

Hence

$$\int_c \frac{f'(z)dz}{f(z)} = \int_a^b \frac{r'(t)dt}{r(t)} + i\int_a^b \theta'(t)dt$$

$$= \left[\log r(t)\right]_a^b + i\left[\theta(t)\right]_a^b$$

As $r(b)=r(a)$ and $\theta(b)-\theta(a) = \Delta_c \arg f(z)$

we have

$$\int_c \frac{f'(z)dz}{f(z)} = i\Delta_c \arg f(z)$$

Again from the previous theorem we have

$$\frac{1}{2\pi i}\int_c \frac{f'(z)\,dz}{f(z)} = N - P$$

Consequently we have

$$\Delta_c \arg f(z) = 2\pi(N - P)$$

Example : Consider the function $f(z) = 1/z^2$, and the circle $C : |z| = r,\ r > 0$. $f(z)$ has a pole of order two at $z=0$.
It is analytic and non - zero on C.
Hence by the argument principle

$$\Delta_c \arg f(z) = 2\pi(0-2) = -4\pi_{//}$$

> **Check your progress :**
>
> (k) Find $\Delta_c\ arg\ f(z)$, where $f(z) = \frac{\sin z}{z(z-1)^2}$ and $C : |z| = 2$

2.5.2 Rouche's Theorem

We will use argument principle to prove a result, which compare the number of zeros of two analytic functions inside a simple closed contour.

***Theorem1* (Rouche's theorem):** Let C be a simple closed contour. Suppose *f* and g are two functions analytic within and on C, such that $|f(z)| > |g(z)|$ for all z on C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities inside C. [i.e same zero may occur move than once and they are counted according to the number of times they occur].

Proof : Since $\left|f(z)\right| > \left|g(z)\right| \geq 0$

and $\left|f(z)+g(z)\right| \geq \left\|f(z)\right| - \left|g(z)\right\| > 0$, $\forall z$ on C.

neither *f*(*z*) nor $f(z)+g(z)$ has a zero on C. Let Z_f and Z_{f+g} denote the number of zeros (counting multiplicities), of *f*(*z*) and $f(z)+g(z)$ respectively inside C.

Then by Argument Principle

$$\Delta_c \arg f(z) = 2\pi Z_f \text{ and } \Delta_c \arg\left[f(z)+g(z)\right] = 2\pi Z_{f+g}$$

Now,

$$\Delta_c \arg\left[f(z)+g(z)\right] = \Delta_c \left\{ f(z)\left[1+\frac{g(z)}{f(z)}\right]\right\}$$

$$= \Delta_c \arg f(z) + \Delta_c \arg\left[1+\frac{g(z)}{f(z)}\right]$$

$$\therefore 2\pi Z_{f+g} = 2\pi Z_f + \Delta_c \arg F(z)$$

where $F(z) = 1+\dfrac{g(z)}{f(z)}$

But $\left|F(z)-1\right| = \left|\dfrac{g(z)}{f(z)}\right| < 1$

That means under the transformation *F*, the image of *C* lies within the open disk $\left|w-1\right| < 1$. That is the image curve does not enclose the origin *w*= *0*. Hence $\Delta_c \arg F(z) = 0$.

Consequently

$$2\pi Z_{f+g} = 2\pi Z_f$$

$$\Rightarrow Z_{f+g} = Z_f$$

Example : Locate the number of zeros of the polynomial $z^7 - 4z^3 + z - 1$

Let $f(z) = -4z^3$ and $g(z) = z^7 + z - 1$

and let C be the circle $|z| = 1$.

Then $|f(z)| = 4$ and $|g(z)| = |z^7| + |z| + 1 \le 3$ on C. Conditions of Rouche's theorem are valid. So, we have $f(z)$ and $f(z) + g(z)$ have same number of zeros inside C. But $f(z)$ has three zeros inside C. So, $f(z)+g(z)$ i.e. the polynomial $z^7 - 4z^3 + z - 1$ has three zeros inside C.

> **Check your progress**
>
> (1) Find the number of zeros of the polynomial $z^6 - 5z^4 + z^3 - 2z$ inside the circle $|z| = 1$.

Next we will prove the fiundamental theorem of algebra using Rouche's theorem.

Let

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n, \quad a_n \ne 0$$

Let $f(z) = a_n z^n$ and $g(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{n-1} z^{n-1}$

We have

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|a_0 + a_1 z + ... + a_{n-1} z^{n-1}|}{|a_n||z|^n}$$

$$\le \frac{|a_0| + |a_1||z| + |a_2||z|^2 + ... + |a_{n-1}||z|^{n-1}}{|a_n||z|^n}$$

Then for all $\forall z : |z| = R$

$$\left|\frac{g(z)}{f(z)}\right| \le \frac{|a_0| + R|a_1| + R^2|a_2| + ... + R^{n-1}|a_{n-1}|}{|a_n|R^n}$$

$$< \frac{|a_0| + |a_1| + |a_2| + ... + |a_{n-1}|}{|a_n|R}$$

By taking R sufficient large we can have

$$\left|\frac{g(z)}{f(z)}\right| < 1 \text{ is } |g(z)| < |f(z)| \ \forall z : |z| = R$$

Let C be the circle $|z| = R$.

Then conditions of Rouche's theorem are valid for $f(z)$, $g(z)$ and C. Hence $f(z)$ and $f(z)+g(z)$ have same number of zeros inside C. As $f(z) = a_n z^n$ has n zeros inside C, so $f(z)+g(z)$ i.e. $P(z)$ has n zeros inside C.

Example : Show that three roots of $z^4 + 6z + 1 = 0$ lies in the annulus $3/2 < |z| < 2$

Solution : Let $f(z) = z^4$ and $g(z) = 6z+1$

and let $C : |z| = 2$

Then on C, $|f(z)| = 32$ and $|g(z)| \le 6|z| + 1 \le 13$

i.e. $|f(z)| > |g(z)|$ on c.

As $f(z)$ has 4 zeros inside, $f(z)+g(z)$ has 4 zeros inside C.

Next let $f(z)=6z$ and $g(z) = z^4 + 1$ and let $c_1 : |z| = 3/2$

then on c_1, $f(z) = 9$ and $|g(z)| = |3|^4 + 1 < 7$

so, $|f(z)| < |g(z)|$ for z on c_1.

As $f(z)$ has single zero inside c_1,

$f(z)+g(z)$ has single zero inside c_1

Thus the number of roots of $z^4 + 6z + 1 = 0$ in the annulus $3/2 < |z| < 2$ is $(4-1)$ is 3.

Check your progress :

(m) Determine the number of roots of

$2z^5 - 6z^2 + z + 1 = 0$

in the annulus $1 < |z| < 2$.

2.5.3 **Evaluation of Real Integrals**

Here we will develop technique to evaluate two types of real integrals :

(a) Improper integral and

(b) Definite integral involving *sines* and *cosines*

(a) Improper integral

In real calculus, the improper integral of a continuous function $f(x)$ on $[0,\propto]$ is defined by

$$\int_0^{\propto} f(x)\,dx = \underset{R\to\propto}{Lt}\int_0^R f(x)\,dx \text{ .... (i)}$$

provided the limit exists. Then the improper integral is said to converge. If $f(z)$ is continuous on $(-\propto,\propto)$. we define

$$\int_{-\alpha}^{\propto} f(x)\,dx = \underset{R_1\to-\propto}{Lt}\int_{R_1}^{0} f(x)\,dx + \underset{R_2\to\propto}{Lt}\int_{R_2}^{0} f(x)\,dx \text{ .... (ii)}$$

whenever both integrals on the R.H.S exists. We have another value, called Cauchy principle value, defined by

$$PV\int_{-\propto}^{\propto} f(x)\,dx = \underset{R\to\propto}{Lt}\int_{-R}^{R} f(x)\,dx \text{ .... (iii)}$$

provided the limit exists.

If integral (ii) converges its Cauchy principle value (iii) exists, and that value is that to which integral (ii) converges. This is because

$$\int_{-R}^{R} f(x)\,dx = \int_{-R}^{0} f(x)\,dx + \int_{0}^{R} f(x)\,dx$$

and the limit as $R\to\propto$ of each of the integrals on the right exists when integral (ii) converges. The converse, however is not true as is evident from the following example.

Example 1 : Consider the function $f(x) = x$.
then

$$\text{PV}\int_{-\infty}^{\infty} f(x)\,dx = \underset{R\to\infty}{Lt}\int_{-R}^{R} x\,dx = \underset{R\to\infty}{Lt}\left[\frac{x^2}{2}\right]_{-R}^{R} = 0$$

Now,

$$\int_{-\infty}^{\infty} f(x)\,dx = \underset{R_1\to\infty}{Lt}\int_{-R_1}^{0} x\,dx + \underset{R_2\to\infty}{Lt}\int_{0}^{R_2} x\,dx$$

$$= \underset{R_1\to\infty}{Lt}\left[\frac{x^2}{2}\right]_{-R_1}^{0} + \underset{R_2\to\infty}{Lt}\left[\frac{x^2}{2}\right]_{0}^{R_2}$$

$$= \underset{R_1\to\infty}{Lt}\frac{R_1^2}{2} + \underset{R_2\to\infty}{Lt}\frac{R_2^2}{2}$$

These two limits do not exist.

Hence the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ does not exists.

It is interesting to note that for even functions, (i.e. for functions satisfying $\left(f(-x) = f(x)\right)$, the converse is also true. Also in that case, the integrals (ii) and (iii) converge or diverge together and

$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} f(x)\,dx$$

We will consder rational functions only

$$\text{Let } f(z) = \frac{p(z)}{q(z)}$$

We assume $q(z)$ has no real zero and at least one zero above the real axis.

We consider a circle $|z|=R$ such that all zeros of $q(z)$ above the real axis are contained inside the closed path formed by the line segment $-R \le x \le R$ and the upper semi circle. If the upper semi-circle is denoted by C_R. then by Cauchy residue theorem, we have

$$\int_{-R}^{R} f(x)\,dx + \int_{c_R} f(x)\,dx = 2\pi i \text{ [sum of the residues of singularities inside the contour]}$$

or,

$$\int_{-R}^{R} f(x)\,dx = 2\pi i \text{ [sum of the Residues] } - \int_{c_R} f(x)\,dx$$

If $\underset{R\to\infty}{Lt} \int_{C_R} f(x)\,dx = 0$, then

$$PV\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \quad \text{[Sum of Residues]}$$

Since $f(z)$ is assume to be even,

$$\int_{-\infty}^{\infty} f(x)\,dx = PV\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \text{ [sum of the residues]}$$

$$\Rightarrow 2\int_{0}^{\infty} f(x)\,dx = 2\pi i \text{ [sum of the residues]}$$

$$\Rightarrow \int_{0}^{\infty} f(x)\,dx = \pi i \text{ [sum of the residues]}$$

While evaluating improper integral by this method, the important fact is toverify that

$$\underset{R\to\infty}{Lt} \int_{c_R} f(x)\,dx = 0$$

Example 2: Evaluate $\int_0^{\infty} \frac{x^2 dx}{x^6+1}$

consider the function $f(z) = \frac{z^2}{z^6+1}$

The singularities of $f(z)$ are the zeros of z^6+1.

these are given by

$$z_k = e^{i(2k+1)\pi/6}, \text{ (k = 0, 1, 2, ..., 5)}$$

the singularities lying above real axis are

$$c_0 = e^{i\pi/6}, \qquad c_1 = e^{i3\pi/6} = i, \quad c_2 = e^{i5\pi/6}$$

Let C_R denote the upper half of the circle $|z| = R$ $(R>1)$. then the singularities c_0, c_1, c_2 lie within the simple closed contour formed by the line segment $-R \le x \le R$ and C_R.

We have

$$\int_{-R}^{R} f(x)\,dx + \int_{c_R} f(z)\,dz = 2\pi i\,[\text{sum of residues at } c_0, c_1 \text{ and } c_2]$$

Using methods to find residue, we obtain the residues as $1/6i$, $-1/6i$ and $1/6i$ respectively.

Hence

$$\int_{-R}^{R} f(x)\,dx = 2\pi i\left(1/6i - 1/6i + 1/6i\right) - \int_{c_R} f(z)\,dz$$

$$= \pi/3 - \int_{c_R} f(z)\,dz$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = PV\int_{-\infty}^{\infty} f(x)\,dx = \underset{R\to\infty}{Lt}\int_{-R}^{R} f(x)\,dx = \pi/3 - \underset{R\to\infty}{Lt}\int_{c_R} f(z)\,dz$$

Finally we need to show $\underset{R\to\infty}{Lt}\int_{c_R} f(z)\,dz = 0$

we have for any z on C_R

$$|f(z)| = \frac{|z|^2}{|z^6+1|} \le \frac{|z|^2}{|z|^6 - 1} = \frac{R^2}{R^6-1} = M_R \text{ (say)}$$

then $\left|\int_{c_R} f(z)\,dz\right| \le M_R \pi R$, πR being length of C_R.

But $M_R \pi R = \pi R \dfrac{R^2}{R^6-1} = \dfrac{\pi R^3}{R^6-1} \to 0$ as $R \to \propto$

Hence $\underset{R\to\propto}{Lt} \int_{c_R} f(z)\,dz = 0$

Consequently $\int_{-\propto}^{\propto} f(x)\,dx = \pi/6$

$$\Rightarrow \int_0^{\propto} f(x)\,dx = \pi/3$$

Example 3 : Evaluate $\int_0^{\propto} \dfrac{x^2}{x^4+1}\,dx$

consider the function $f(z) = \dfrac{z^2}{z^4+1}$

the singularities of $f(z)$ are the zeros of z^4+1

then are given by $z_k = e^{i(2k+1)\pi/4}$, (k = 0, 1, 2, 3)

i.e. $e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$ and $e^{i7\pi/4}$

The singularities lying above the x-axis are $z_1 = e^{i\pi/4}$ and $z_2 = e^{i3\pi/4}$

The residues at z_1 and z_2 are $\dfrac{1-i}{4\sqrt{2}}$ and $-\dfrac{1+i}{4\sqrt{2}}$

Hence

$$\int_{-R}^{R} f(x)\,dx + \int_{c_R} f(z)\,dz = 2\pi i\left(\frac{1-i}{4\sqrt{2}} - \frac{1+i}{4\sqrt{2}}\right)$$

$$\Rightarrow \int_{-R}^{R} f(x)\,dx = \frac{\pi}{\sqrt{2}} - \int_{c_R} f(z)\,dz$$

Hence

$$\int_{-\infty}^{\infty} f(x)\,dx = PV\int_{-\infty}^{\infty} f(x)\,dx = \underset{R\to\infty}{Lt}\int_{-R}^{R} f(x)\,dx$$

$$= \frac{\pi}{\sqrt{2}} - \underset{R\to\infty}{Lt}\int_{c_R} f(z)\,dz\ .$$

Here C_R denote the upper half of the circle of the circle $|z| = R(>1)$

Finally show $\underset{R\to\infty}{Lt}\int_{c_R} f(z)\,dz = 0$

we have for any z on $|z| = R$,

$$|f(z)| = \frac{|z|^2}{|z^4+1|} \le \frac{|z|^2}{|z|^4-1} = \frac{R^2}{R^4-1} = M_{\mathbf{R}}\text{ (say)}$$

then $\left|\int_{c_R} f(z)\,dz\right| \le M_R \pi R$, πR being the length of C_R

But $M_R \pi R = \frac{R^2}{R^4-1}\pi R = \pi\frac{R^3}{R^4-1} \to 0$ as $R \to \infty$

Consequently

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow \int_{0}^{\infty} f(x)\,dx = \frac{\pi}{2\sqrt{2}}$$

Check your progress

(n) Evaluate

(i) $\int_0^{\infty} \frac{dx}{x^2+1}$

(b) **Definite integral involving *sines* and *cosines***

The method of residues is also useful in evaluating certain definite integrals of the type.

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta, \quad \int_{-\pi}^{\pi} f(\sin\theta, \cos\theta)\,d\theta \quad \text{...........(i)}$$

Here θ is the argument of a point z on the unit circle centred at the origin.

In these type of problems, we substitute

$$z = e^{i\theta} \; (0 \le \theta \le 2\pi \text{ or } -\pi \le \theta \le \pi)$$

which gives $dz = ie^{i\theta} d\theta = iz d\theta$

Again the relations

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

i.e. $\sin\theta = \dfrac{z - z^{-1}}{2i}$, $\cos\theta = \dfrac{z + z^{-1}}{2}$

reduces the given integrals to the form

$$\int_c f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{iz} \quad \text{................... (ii)}$$

where $c : |z| = 1$ with positive direction. If the integrand of integral (ii) is a rational function of z, we can evaluate that integral by means of Cauchy's residue theorem once the zeros of the denominator polynomial are known and provided they do not lie on C.

Example 1: Evaluate

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \; (-1 < a < 1)$$

Let $z = e^{i\theta} \; (0 \le \theta \le 2\pi)$, then $dz = izd\theta$

Let C denote the circle $|z| = 1$ with positive direction.

Then the given integral reduces to

$$\int_c \frac{dz}{iz\left(1+a\dfrac{z-z^{-1}}{2i}\right)}$$

$$=\int_c \frac{2/a}{z^2+\dfrac{2i}{a}z-1}dz$$

The zeros of the denominator of the integral are

$$z_1=\left(\frac{-1+\sqrt{1-a^2}}{a}\right)i\,,\quad z_2=\left(\frac{-1-\sqrt{1-a^2}}{a}\right)i$$

As $|a|<1$,

$$|z_2|=\frac{1+\sqrt{1-a^2}}{|a|}>1$$

Also, since $|z_1z_2|=1$, we have $|z_1|<1$.

Hence neither singularities lie on C, and the only singularity interior to it is the point z_1. The residue at z_1 is $\dfrac{1}{i\sqrt{1-a^2}}$.

Consequently

$$\int_c \frac{2/a}{z^2+\dfrac{2i}{a}z-1}dz=2\pi i\frac{1}{i\sqrt{1-a^2}}=\frac{2\pi}{i\sqrt{1-a^2}}$$

Example 2 : Evaluate

$$\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2}\,,\quad a^2>1$$

The integrand is symatric in θ, so we write

$$\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2}=\frac{1}{2}\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2}\,,\quad a^2>1$$

Put $z=e^{i\theta}$ and $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$

then $dz = izd\theta$ and $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Let C denote the unit circle $|z| = 1$ with positive direction. then the given integral reduces to

$$\int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{4}{i}\int_c \frac{zdz}{(z^2+2az+1)^2}$$

the singularities of the integrand are poles of order two at $z_1 = -a + \sqrt{a^2-1}$ and $z_2 = -a - \sqrt{a^2-1}$

As $a^2 > 1$,

$$|z_2| = \left|a + \sqrt{a^2-1}\right| > 1$$

Also, since $|z_1 z_2| = 1$, we have $|z_1| < 1$

Hence neither singularities lie on C, and the only singularity interior to it is z_1. The residue at z_1 is $\frac{a}{4(a^2-1)^{3/2}}$

Consequently

$$\frac{4}{i}\int_c \frac{zdz}{(z^2+2az+1)^2} = \frac{4}{i} \text{x} 2\pi i \frac{a}{4(a^2-1)^{3/2}}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} = 2\pi \frac{a}{(a^2-1)^{3/2}}$$

$$\Rightarrow \int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{i\pi a}{(a^2-1)^{3/2}}$$

Check your progress :

(o) Evaluate

(i) $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$, (ii) $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$

2.7 1. Let us sum up :

In this blockt you are acquainted with two important concepts : Taylor expansion and Laurent expansion of a function, Whenever a function is analytic at a point then there is always a certain neighbourhood of that point where the function has a power series expansion. If f has an isolated singularity at some point then in some deleted neighbourhood of that point, the function has a series representation. One important conclusion that we have come across is that zeros of analytic function are isolated. The concept of Laurent series helps to classify different kinds of isolated singularities of a function. Beside this theorem, we have developed some other methods to evaluate tesidues at singularities. If a function is analytic within and on a smple closed contour C with known number (finit) of poles inside C, then Arguments principle determines the number of zeroes of f inside C. Rouche's theorem on the other hand, compares the number of zeroes of two functions related by a certain condition. the most important application of Cauchy residue theorem is the evaluation of some real integrals.

2. Keywords :

Zeros of analytic function, Singularities, poles, Taylor Series, Laurent Series, Residues, Arguments Principle, Rouche's Theorem.

3. References :

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3. H.S. Kasana, Complex Variable, Theory and Application Prentice Hall of India.

4. Possible Answers to the CYP :

(a) Proceed as in Example 1 (b) of 5.2

(b) Proceed as in Example 2 of 5.2

(c) $z^3 \sin z = 0 \Rightarrow z^3 = 0$ or $\sin z = 0 \Rightarrow z = 0$ or $z = n\pi$, $n \in N$

Let $z^3 = f(z)$ and $\sin z = g(z)$

then z=0 is a zero of order 3 for

$f(z)\{\because f(0) = f^{/}(0) = f^{//}(0) = 0, \text{ but } f^{///}(0) \neq 0\}$

and z=0 is a simple zero for $g(z)\{\because g(0) \text{ but } g^{/}(0) \neq 0\}$

So, z=0 is a zero of order (3+1) i.e. 4 for $z^3 \sin z$

Again for $0 \neq n \in N, g(n\pi) = 0$ but $g^{/}(n\pi) \neq 0$

So, $z = n\pi\ (n \neq 0)$ is a zero of order one for $g(z)$ and hence for $f(z)g(z)$ is $z^3 \sin z$.

(d) (i) $f(z) = 1 + z^2 = 0 \Rightarrow z = \pm i$

$f^{/}(z) = 2z,\ f^{//}(z) = 2$

$\therefore f(i) = 0, f^{/}(i) \neq 0,\ \therefore z = i$ is a simple zero for $1+z^2$ Similarly $z=-i$ is a simple zero for $1+z^2$

(ii) Proceed as in d(i).

(e) (i) $$\frac{z - \sin z}{z^4} = \frac{1}{z^4}\left[z - \left\{\frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \ldots\right\}\right]$$

$$= \frac{1}{z3!} + \frac{z^2}{5!} - \frac{z^3}{7!} \ldots, \qquad 0 < |z| < \propto$$

(ii) Proceed as in (e) (i).

(f) (i) Let $f(z) = \frac{1}{1+z^2}$. Find the laurent Expansion of $\frac{1}{z^2} f(z)$ about z=0 than use theorem 2 of section 5.4.1

(ii) Proceed as in f(ii)

(g) (i) $$\frac{1}{z(z+1)} = \frac{1}{z}(1+z)^{-1}$$

$$= \frac{1}{z}\left(1 - z + z^2 - z^3 + ...\right),\ 0 < |z| < 1$$

$$= \frac{1}{z} - 1 + z - z^2 + ...,\ 0 < |z| < 1$$

This is the Laurent Series of $\frac{1}{z(z+1)}$ about z=0 .

Residue at z=0 is coefficient of $1/z$ i.e. 1

Again, $$\frac{1}{z(z+1)} = -\frac{1}{(z+1)(1-(z+1))}$$

$$= -\frac{1}{(z+1)}\left\{1-(z+1)\right\}^{-1}$$

$$= -\frac{1}{(z+1)}\left\{1 + (z+1) + (z+1)^2 + ...\right\},$$

$$0 < |z+1| < 1$$

$$= -\frac{1}{z+1} - 1 - (z+1) - ...,\ 0 < |z+1| < 1$$

This is the Laurent series of $\frac{1}{z(z+1)}$ about z=-1.

Residue at z=*-1* is coefficient of $\frac{1}{(z+1)}$ i.e. 1

for (g) (ii) and (iii) Use the series of $\cos 1/z$ and $\sin z$.

(h) (i), (ii) and (iii) Proceed as in examples 1 & 2 given after Method 2.

(i) (i), (ii) and (iii) Proceed as in examples 1 & 2 given after Method 3.

(j) Find the number of zeros and poles of $f(z)$ inside $|z|=3$, then apply theorem 1 of section 5.5.1.

(k) Find the number of zeros and poles of $f(z)$ inside $|z|=3$, then apply theorem 2 of section 5.5.1.

(l) Take $f(z)=5z^4$ and $g(z)=z^6+z^3-2z$. Then use Rouche's theorem.

(m) First take $f(z)=-6z^2$ and $g(z)=2z^5+z+1$. Then Using rouche's theorem show that the given equation has two roots inside $|z|<1$.

Next take $f(z)=2z^5$ and $g(z)=-6z^2+z+1$ and using Rouches theorem show that the given polynomial has five roots inside $|z|<2$.

(n) Let $f(z)=\dfrac{1}{z^2+1}$. Then $z=-i\,\&\,i$ are the singularities of $f(z)$. The singularity above x-axis is i.

Find Radue of $f(z)$ at $z=i$. It is $\dfrac{1}{2i}$. Then proceed as in examples 2 and 3 of section 5.5.3 (a)

(o) Put $z=e^{i\theta}$ and $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$, and then proceed as in example 1 and 2 section 5.5.3 (b)

5. Model Questions :

1. Find the Maclaurin series expansion of

(a) $\sin z^2$, (b) $z\cosh^2 iz$, (c) e^{e^z}

2. Obtain Taylor series of

(a) $\cosh 2z$ about $z=i\pi$

(b) $\sin z$ about $z=2$

3. Derive the expansion

$$\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!},\ 0 < |z| < \infty$$

4. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$$

5. Locate the zeros of the function and determine their order

(a) $(1+z^2)^4$, (b) $z^8 + z^4$

6. If f has zero of order k at a, show that f' has a zero of order $(k-1)$ at a.

7. Obtain the Laurent series expansion of

$$f(z) = \frac{z}{(z-1)(z-3)}$$

8. Find the Laurent series that represents the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$

9. Find a representation for the function

$$f(z) = \frac{1}{1+z} = \frac{1}{z} - \frac{1}{1 + \frac{1}{z}}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

10. Find different series expansions in certain domains and specify those domains for the function

$$f(z) = \frac{1}{z(1+z^2)}$$

11. In each case, write the principal part of the function at its isolated singular point and classify the singularity

(a) $ze^{1/z}$, (b) $\dfrac{z^2}{1+z}$, (c) $\dfrac{\sin z}{z}$,

(d) $\dfrac{\cos z}{z}$, (e) $\dfrac{1}{(2-z)^3}$

12. Show that the singular point of each of the following functions is a pole. Determine the order of the pole and the corresponding residues.

(a) $\dfrac{1-\cosh z}{z^3}$, (b) $\dfrac{1-e^{2z}}{z^4}$,

(c) $\dfrac{e^{2z}}{(z-1)^2}$

13. Suppose that a function f is analytic at z_0, and write $g(z)=\dfrac{f(z)}{z-z_0}$

show that –

(a) If $f(z_0)\neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$.

(b) If $f(z_0)=0$, then z_0 is a removable singular point of g.

14. Evaluate thef integrals using residue theorem :

(a) $\displaystyle\int_c \frac{3z^2+2}{(z-1)(z^2+9)}dz$ when $c:|z-2|=2$

(b) $\displaystyle\int_c \frac{dz}{z^3(z+4)}$ when $c:|z|=2$

(c) $\displaystyle\int_C \frac{\cosh \pi z dz}{z(z^2-1)}$ when $c:|z|=2$

15. Evaluate using single residue method

(a) $\int_c \frac{\left(3z^2+2\right)^2}{z(z-1)(2z+5)}dz$, (b) $\int_c \frac{z^3 e^{1/z} dz}{1+z^3}$

where $c:|z|=3$.

16. Let c denote the unit circle $|z|=1$. Find $\Delta_c \, arg\, f(z)$ where

(a) $f(z)=z^2$, (b) $f(z)=\frac{z^3+2}{z}$, (c) $\frac{(2z-1)^7}{z^3}$

17. Determine the number of zeros, counting multiplicities of the polynomial

(a) $z^6-5z^4+z^3-2z$ inside the circle $|z|=1$

(b) $z^4-2z^3+9z^2+z-1$ inside the circle $|z|=2$

18. Show that if c is a complex number such that $|c|>e$, then the equation $cz^n = e^z$ has n roots, counting multiplications, inside the circle $|z|=1$.

19. Let $P(z)=4z^2-3iz^2+iz-9$,

Use Rouchi's theorem to show that there are

(a) no zero in $|z|<1$, (b) three zero in $|z|<2$.

20. Use residues to evaluate the improper integrals

(a) $\int_0^\infty \frac{dx}{\left(x^2+1\right)^2}$, (b) $\int_0^\infty \frac{x^2 dx}{\left(x^2+1\right)\left(x^2+4\right)}$

(c) $\int_{-\infty}^{\infty} \frac{\left(x^2+x+1\right)dx}{\left(x^2+1\right)\left(x^2+4\right)^2}$

21. Use residues to evaluate the definite integrals :

(a) $\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{5-4\cos 2\theta}$, (b) $\int_{-\pi}^{\pi} \frac{d\theta}{\left(1+\sin^2\theta\right)^2}$, (c) $\int_{-\pi}^{2\pi} \frac{1+\sin\theta}{3+\cos\theta}d\theta$

UNIT-3 : CONFORMAL MAPPING [12 Marks]

STRUCTURE :

6.0 **Objectives**

- After going through this unit you will be able to
- Define and give examples of Bilinear transformation.
- Examine whether a given transformation is a Bilinear
- Prove Properties of Bilinear Transformation
- Find fixed point(s) of a transformation
- Find Image under a given bilinear transformation
- Construct bilinear transformation carrying specified points to specified points.

- v Obtain the mapping of the upper half plane
- v Define and give examples of isolgonal and conformal mapping
- v Prove necessary condition for a function to be conformal
- v Identify points where a function is conformal
- v Find scalar factor and angle of rotation of a given function.

6.1 Introduction

In this unit we will discuss different types of transformatiom. Some elementry transformations are translation, rotation, stretching (contraction), and Inversion. A composition of a stretching (Contraction) and a translation is called a linear transformation. You will be introduced to the new concept of Bilinear transformation.

A bilinear transformation is a combination of translation, rotation, stretching (contraction), and inversion. Some important properties of bilinear transformation will be discussed. We will obtain the bilinear transformation that maps the upper half-plane onto the open unit disk. The Bilinear transformations fall into a broader class of transformations called conformal mapping or angle preserving mapping. We will obtain sufficient condition for a function to be conformal and discuss some of its properties. Two important quantities associated with a function are the scalar factor and angle of rotation. You will learn how to calculate these quantities. Finally we will show that a function, which is conformal at a point, has a local inverse at that point.

6.2 Some elimentary transformation

Translation : A transformation of the form $w=z+a$, where a is some constant complex number, is called a translation

we write, $w = u + iv$ and $z = x + iy$

Suppose $a = a_1 + ia_2$

then $w = z + a \Rightarrow u + iv = (x + iy) + (a_1 + ia_2)$

$$\Rightarrow u = x + a_1,\ v = y + a_2$$

Thus the image of any point (x, y) in the z-plane is the point $(u, v) = (x + a_1, y + a_2)$ in the w-plane. Hence any point under this transformation is translated though a distance $|a|$ in the direction of the vector $\bar{a} = (a_1, a_2)$ since this is simply a translation image of any region is geometrically congruent to the original one.

Check your progress :

(a) Locate geometrically the image of the region : $|z| \leq 1$ under the transformation $w = z + 2 + i$

Rotation : A transformation of the form $w = e^{i\alpha} z$, where α is some real number, is called a rotation

Write $w = e^{i\alpha} z$, $z = re^{i\theta}$, then

$$w = e^{i\alpha} z \Rightarrow \mathrm{Re}^{i\phi} = e^{i\alpha} re^{i\theta}$$

$$\Rightarrow \mathrm{Re}^{i\phi} = re^{i(\alpha+\theta)}$$

$$\Rightarrow R = r,\ \phi = \alpha + \theta$$

Hence any point under this transformation is rotated through an angle. Since this is simply rotation, image of any region is geometrically congruent to the original one.

Check your Progress :
(b) Repeat CYP (a) under the transformation $w = e^{i\pi/4} z$

Contraction or Strecthing : A transformation of the form $w = az$, where a is some complex constant, is called contraction (if $|a|<1$) or stretching (if $|a|>1$)

Write $w = \text{Re}^{i\phi}$, $z = re^{i\theta}$,

suppose $a = \rho e^{i\alpha}$, where ρ and α an real constants.

Then $\quad w = az \Rightarrow \text{Re}^{i\phi} = \rho e^{i\alpha} = \rho e^{i(\alpha+\theta)}$

$\Rightarrow R = fr, \qquad \phi = \alpha + \theta$

Thus under this transformation, the radius vector of any point is stretched (if $\rho > 1$) or contracted (if $\rho < 1$) by the factor ρ. Also it is rotated through an angle of α. Since stretching or contraction is involved, the image of any region will not be geometrically congruent to the original one. How ever the geometric shape of the image will be similar to the original one.

Check your progress :	
(c)	Locate geometrically the image of the region $\leq |z| \leq 2$ under the transfofrmation $w = (1+i)z$

Linear Transformation : A transformation of the form $w = az + b$ where a and b are complex constants, is called a linear transformation.

It is the composition of the taransformation.

$z \rightarrow az$ and $z \rightarrow z + b$

Note : Contraction/Streching is a particular case of linear transformation.

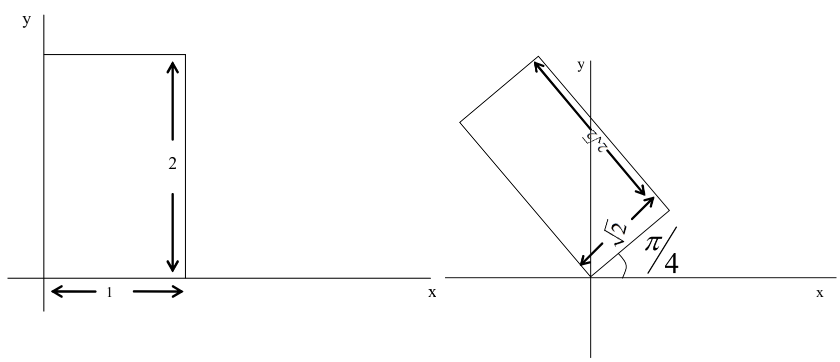
Exercise : Find the image of the region $0 \leq x \leq 1$, $0 \leq y \leq 2$ under the transformation $w = (1+i)z + 2$

Solution: This is a linear transformation. It is the composition of the transformation .

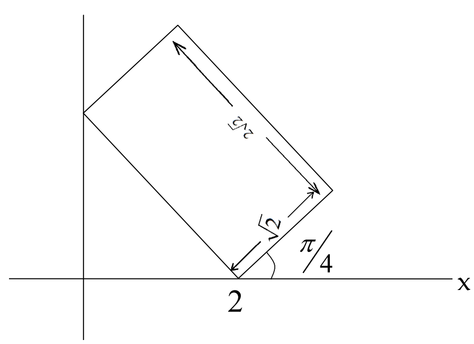
$$w = (1+i)z \text{ and } W = Z + 2$$

We have $$(1+i) = \sqrt{2}e^{i\pi/4}$$

So, under the transformation $w=(1+i)z$ the given region will be expanded by the factor $\sqrt{2}$ and rotated through on angle $\pi/4$. The image is shown as below.



Then under the transformation $w=z+2$. The image region is further translated through a distance of 2 unit horizontally. So final image of the given region is



Check your progress :

(d) Find the region $\{z : z = 2e^{i\theta}, 0 \leq \theta \leq \pi\}$, under the transformation $w=iz+1$.

Inversion : A transformation of the form $w = 1/z$ is called an inversion.

Write $w = Re^{i\theta}$ and $z = re^{i\theta}$

then $$w = 1/z \Rightarrow Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$$

$$\Rightarrow R = 1/r \text{ and } \phi = -\theta$$

Hence under this transformation the radius vector of any point is changed to its reciprocal and the point is rotated through an angle $-\theta$. The map establishes a one-one corresponding between the non-zero point of the z and w-planes. Since $|w| = \frac{1}{|z|} \Rightarrow |w||z| = 1$, the interior of the circle $|z| = 1$ is mapped onto the exterior of the circle $|w| = 1$, $|z| = 1$ is mapped on $|w| = 1$, and the enterior of the circle $|z| = 1$ is mapped onto interior of the circle $|w| = 1$.

The transformation can be extended to the extended complex plane $C \cup \{\propto\}$ as

$$w = \begin{cases} 0, z = \propto \\ \propto, z = 0 \\ 1/z, \text{otherwise} \end{cases}$$

With this definition Tz=w is a continuous function in the extended complex plane.

writing $w = u + iv$ and $z = x + iy$ in the equation $w = 1/z$

we get $u = \frac{x}{x^2 + y^2}$, $v = -\frac{y}{x^2 + y^2}$ .... (i)

and $x = \frac{u}{u^2 + v^2}$, $y = -\frac{v}{u^2 + v^2}$ .... (ii)

Equation (i) and (ii) will be used to find images of different region under the transformation $w = 1/z$.

Result : The map $w = \frac{1}{z}$ transforms circles and lines into circles and lines.

Proof : Let a, b, c, d be real numbers such that $b^2 + c^2 > 4ad$. Then the equation

$$a\left(x^2 + y^2\right) + bx + cy + d = 0 \quad \text{..........(iii)}$$

represents a circle if $a \neq 0$ and a line if $a=0$. Using (ii), (iii) reduces to

$$d\left(u^2 + v^2\right) + bu - cv + a = 0 \quad \text{............ (iv)}$$

It is a circle (in the w-plane if $d \neq 0$ and a straight line of d=0. From (iii) and (iv) it is obvious that –

1. A circle not passing through the origin $(a \neq 0, d \neq 0)$ in the z-plane is mapped onto a circle in the w-plane not passing through the origin.

2. A circle passing through the origin $(a \neq 0, d = 0)$ in the z-plane is mapped onto a line in the w-plane not passing through the origin.

3. A line not passing through the origin $(a = 0, d \neq 0)$ in the z-plane is mapped onto a circle in the w-plane passing through the origin.

4. A line passing through the origin $(a = 0, d = 0)$ in the z-plane is mapped onto a line in the w-plane passing through the origin.

Example 1 : When a circle is transformed into a circle under the map $w = \frac{1}{z}$, the centre of the original circle is never mapped onto the centre of the image circle.

Solution : From (iii) and (iv) of the above result we have centre of the original circle is $\left(-\frac{b}{2a},-\frac{c}{2a}\right)$ and the centre of the image circle is $\left(-\frac{b}{2d},\frac{c}{2d}\right)$. Again, the image of $\left(-\frac{b}{2a},-\frac{c}{2a}\right)$, the centre of the original circle under $w=1/z$ is given by

$$u=\frac{-b/2a}{\left(b^2+c^2\right)/4a^2},\qquad v=\frac{c/2a}{\left(b^2+c^2\right)/4a^2}\text{, using (i)}$$

i.e. $\quad u=\frac{-2ab}{b^2+c^2},\qquad v=\frac{2ac}{b^2+c^2}$

If we assume that the centre of the circle is mapped onto the centre of the image circle, then

$$\frac{-2ab}{b^2+c^2}=-b/2a\Rightarrow b^2+c^2=4ad$$

and $\quad \frac{2ac}{b^2+c^2}=\frac{c}{2d}\Rightarrow b^2+c^2=4ad$

But of $b^2+c^2=4ad$, (iii) will not represent a circle. Hence the centre of the original circle is not mapped to the centre of the image circle.

Example 2 : Find the image of the infinite strip $0<y<\frac{1}{2c}$ under the map $w=1/z$.

Solution : From (ii) we have under the transformation

$$w=1/z,\qquad x=\frac{u}{u^2+v^2},\qquad y=\frac{-v}{u^2+v^2}$$

So, $\quad 0<y\Rightarrow\frac{-v}{u^2+v^2}\Rightarrow v>0\Rightarrow -r>0\Rightarrow -v>0\Rightarrow v<0$

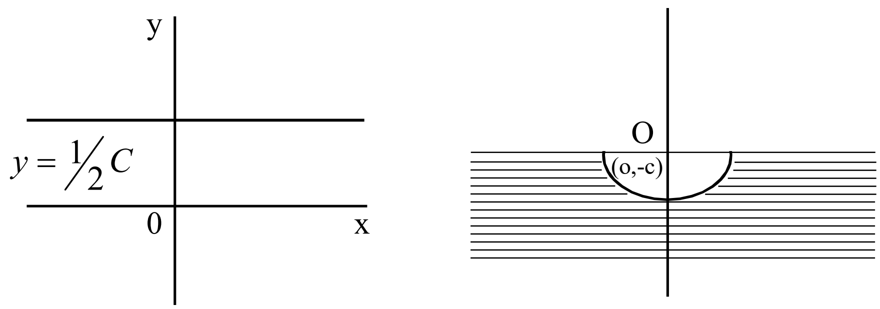
and $\quad y<\frac{1}{2c}\Rightarrow-\frac{v}{u^2+v^2}<\frac{1}{2c}\qquad\Rightarrow u^2+v^2+2cv>0$

$$\Rightarrow u^2+\left(v+c\right)^2>c^2$$

$u^2+(v+c)^2=c^2$ is the circle with centre at (*o,-c*) and radius *c*.

$u^2+(v+c)^2>c^2$ denote the exterior of that circle.

Hence the image of the infinite strip $0<y<\frac{1}{2c}$ is the region shown shaded in the following figure.



Check your progress

(e) Find the image of the half plane $x<c_1\ (c_1<0)$ under the transformation $w=1/z$.

6.3 Bilinear Transformation

6.3.1 Difinition and Example

A transformation of the form

$$W=Tz=\frac{az+b}{cz+d}\ , \quad \text{..................... (i)}$$

where a, b, c, d are complex constant satisfying $ad-bc\neq 0$, is called a bilinear transformation, or Linear fractional transformation or Mobius transformation.

Transformation (i) can also be written as

$cwz+dw-az+b=0$

i.e. $Azw+Bz+Cw+D=0,\ AD-BC\neq 0$

Since it is linear in z and w, it is called bilinear Bilinear transformation is the combination of linear and inverse transformation.

If ad- bc =o in (i), then

$$w = \frac{az+b}{cz+d} = \frac{a\left(z + b/a\right)}{c\left(z + d/c\right)}$$

$$= a/c \, \frac{z + b/a}{z + d/c}, \qquad \because ad - bc = 0 \quad \Rightarrow b/a = d/c$$

$$= a/c$$

which is a constant transformation But a constant function is not linear. Hence we have the condition $ad - bc \neq 0$. You will observe that the behaviour of the transformation will not change if all the constants a, b, c, d are multiplied by a non-zero constant. Hence, without loss of generality we can take *ad - bc = 1.*

Remark : Transformation, Rotation, contraction (or stretching), inversion are all particular cases of bilinear transformation.

The bilinear transformation (i) is not defined for $z = -d/c$ $(c \neq 0)$ in the complex plane. We can however extend the transformation to the extended complex plane by definng it as

$$T' : C_\infty \to C_\infty$$

$$w = T_z' = \begin{cases} \infty, & z = \infty \quad c = 0 \\ a/c, & z = \infty \quad c \neq 0 \\ \infty, & z = -d/c \quad c \neq 0 \\ Tz, & otherwise \end{cases}$$

You can verify that with this definition T'(z) is continuous on the extended complex plane $C_\propto$, Hence T'_Z is one-one onto mapping from $C_\propto$ to $C_\propto$. For simplicity we will write T for T'.

6.3.2 **Properties of bilinear tansformation**

(a) The inverse of a bilinear transformation is bilinear

(b) The composite of two bilinear transformation is bilinear

(c) Every bilinear transformation maps circles and limes into circles and lines.

Proof: (a) Since T is one-one onto T^{-1} exists and given by

$$w = T_z \Leftrightarrow z = T^{-1}w$$

we have $w = T_z = \dfrac{az+b}{cz+d}, \quad ad-bc \neq 0$

Simplifying we get

$$z = \frac{-dw+b}{cw-a}, \quad ad-bc \neq 0$$

$$\therefore T^{-1}w = \frac{-dw+b}{cw-a}, \quad ad-bc \neq 0$$

which is again bilinear

Further $\quad T^{-1}(\propto) = \propto, \text{if } c = 0$

$$= -\frac{d}{c} \text{ if } c \neq 0$$

$$T^{-1}\left(a/c\right) = \propto, \; c \neq 0$$

(b) Let us consider two bilinear transformation

$$T_z = \frac{az+b}{cz+b}, \quad ad-bc \neq 0$$

and $S_z = \dfrac{pz+q}{rz+s}$, $ps - qr \neq 0$

Then

$$(ST)_z = S(T_z) = S\left(\frac{az+b}{cz+d}\right)$$

$$= \frac{p\dfrac{az+b}{cz+d}+q}{r\dfrac{az+b}{cz+d}+s}$$

$$= \frac{(pa+qc)z+(bp+dq)}{(ar+cs)z+(br+ds)}$$

$$= \frac{Az+B}{Cz+D} \qquad \text{(say)}$$

You can easily show that $AD - BC \neq 0$ Hence ST is a bilinear transformation Similarly TS is a bilinear transformation.

(c) Let Tz be a bilinear transformation given by

$$T_z = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

If c = 0, then

$$T_z = \frac{a}{d}z + b/d = Az + B, \quad A = \frac{a}{d}, B = b/d$$

which is a linear transformation we have already seen that a linear transformation maps lines and circles into lines and circles.

If $c \neq 0$, then we can write

$$T_z = \frac{a}{c} + \frac{bc-ad}{c}.\frac{1}{cz+d}$$

Let $cz+d = z_1, \dfrac{1}{z_1} = \dfrac{1}{cz+d} = z_2 \ (say), \quad z_3 = \dfrac{bc-ad}{c}.z_2$

then $T_z = \dfrac{a}{c} + z_3$

i.e., Tz is the composition of the following transformation

$$Z = cz + d\,,\; W = \frac{1}{z}\;,\; w = a/c + \frac{bc - ad}{c}W$$

i.e Tz is the composition of a linear transformation, then an inversion, followed by another linear transformation. But each of the component transformation maps circles and lines into cicles and lines. Hence Tz maps circle and lines into circles and lines.

Example: The line 3y=x is mapped onto a circle under the bilinear transformation

$$w = \frac{iz + 2}{4z + i}$$

Solution : Put w=u+iv and z=x+iy. Then

$$w = \frac{iz + 2}{4z + i} \text{ gives}$$

$$x = \frac{9u}{16u^2 + (4v-1)^2}\,,\quad y = \frac{4u^2 + 4v^2 + 7u - 2}{16u^2 + (4v-1)^2}$$

Putting these values of x and y in 3y=x, we get

$$u^2 + v^2 + 3/4\,u + 7/4\,v - 1/2 = 0$$

i.e. $\left(u + 3/8\right)^2 + \left(v + 7/8\right)^2 = \frac{45}{32}$

It is a circle with centre at $\left(-3/8, -7/8\right)$ and radius equal to $\frac{3}{4}\sqrt{3/2}$

6.3.3 **Some theorems related to Bilinear Transformation :**

Definition **:** A bilinear transformation Tz is said to have a fixed point (or invariant point) at z_0 if $Tz_0 = z_0$

Theorem 1**:** Every bilinear transformation (except the identity map) has at most two fixed point

Proof : We have

$$Tz = z$$

$$\Leftrightarrow \frac{az+b}{cz+d} = z$$

$$\Leftrightarrow cz^2 - (a-d)z - b = 0$$

This being a quaratic equation, will give at most two values of z. Hence Tz has atmost two fixed points. However for the identity map Iz=z, every point is a fixed point.

***Theorem* 2 :** If a bilinear transformation w=Tz has exactly two fixed points, say z_1 and z_2, then for same non-zero constant k,

$$\frac{w-z_1}{w-z_2} = k\frac{z-z_1}{z-z_2}$$

If Tz has only one fixed point, say z_1 then for some non-zero constant k',

$$\frac{1}{w-z_1} = k' + \frac{1}{z-z_1}$$

Proof : Let Tz has exactly two fixed point z_1 and z_2 then

$$Tz_1 = z_1 \text{ and } Tz_2 = z_2$$

$$\Rightarrow \frac{az_1+b}{cz_1+d} = z_1 \text{ and } \frac{az_2+b}{cz_2+d} = z_2$$

$$\Rightarrow cz_1^2 - (a-d)z_1 - b = 0 \text{ and } cz_2^2 - (a-d)z_2 - b = 0$$

$$\Rightarrow cz_1^2 - az_1 = b - dz_1 \text{ and } cz_2^2 - az_2 = b - dz_2 \quad \text{............. (1)}$$

Then

$$w - z_1 = \frac{az+b}{cz+d} - z_1 = \frac{az+b-z_1(cz+d)}{cz+d}$$

$$= \frac{(a-z_1c)z + b - dz_1}{cz+d}$$

$$= \frac{(a - cz_1)z + (cz_1^2 - az_1)}{cz + d} \text{ Using first part of (1)}$$

$$= \frac{(a - cz_1)(z - z_1)}{cz + d}$$

Similarly

$$w - z_2 = \frac{(a - cz_2)(z - z_2)}{cz + d}$$

Hence

$$\frac{w - z_1}{w - z_2} = \frac{(a - cz_1)(z - z_1)}{(a - cz_2)(z - z_2)}$$

$$= k\frac{(z - z_1)}{(z - z_2)}, \quad k = \frac{(a - cz_1)}{(a - cz_2)} \neq 0, \quad \because z_1 \neq z_2$$

If Tz has only one fixed point, then say z_1, then the equation Tz=z has exactly one solution.

i.e. $cz^2 - (a - d)z - b = 0$ has only one root.

It is given by $z_1 = \dfrac{a - d}{2c}$

This gives $a - cz_1 = d + cz_1$ .... (ii)

Hence as before, we have

$$w - z_1 = \frac{(a - cz_1)(z - z_1)}{cz + d}$$

$$\Rightarrow \frac{1}{w - z_1} = \frac{cz + d}{(a - cz_1)(z - z_1)}$$

$$= \frac{cz + d + cz_1 - cz_1}{(a - cz_1)(z - z_1)}$$

$$= \frac{c(z - z_1) + (d + cz_1)}{(a - cz_1)(z - z_1)}$$

$$= \frac{c(z-z_1)+(a-cz_1)}{(a-cz_1)(z-z_1)} \text{ Using (ii)}$$

$$= \frac{c}{(a-cz_1)} + \frac{1}{(z-z_1)}$$

$$= k' + \frac{1}{(z-z_1)}$$

Where $k' = \frac{c}{(a-cz_1)} = \frac{2c}{a+d} \neq 0$, putting $z_1 = \frac{a-d}{2c}$

Example : Find the bilinear transformation which have -1 and 1 as fixed points.

Solution : We have from the previous theorem that if $Tz = \frac{az+b}{cz+d}$ has exactly two fixed points, say z_1 and z_2 then

$$\frac{w-z_1}{w-z_2} = k\frac{z-z_1}{z-z_2} \quad \text{where} \quad k = \frac{a-cz_1}{a-cz_2}$$

Here $z_1 = 1$, $z_2 = -1$, $\therefore k = \frac{a-c}{a+c}$

Hence we have

$$\frac{w-1}{w+1} = \frac{a-c}{a+c} \cdot \frac{z-1}{z+1}$$

which on simplification reduces to

$$w = \frac{az+c}{cz+a}, \quad a^2 - c^2 \neq 0$$

Alternative method

Let $Tz = \frac{az+b}{cz+d}$ has two fixed point 1 and -1 then T (1) = 1 and T (-1) = -1 which gives

$$\frac{a+b}{c+d}=1 \text{ and } \frac{-a+b}{-c+d}=-1$$

On simplifying and solving we get

$a = d$ and $b = c$

Hence required transformation is

$$Tz=\frac{az+c}{cz+a}, \quad a^2-c^2 \neq 0$$

Check your progress

(f) Find all bilinear transformation that have i and -i as fixed points

***Theorem 3* :** There exists a unique bilinear transformation which maps three given distinct points z_1, z_2 and z_3 in the extended z-plane onto three specified distinct points w_1, w_2 and w_3 in the extended w- plane respectively

Proof : Consider the expression

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)}=\frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)} \quad \text{.........................} (i)$$

It can be written in the form

$$(w-w_1)(w_2-w_3)(z_1-z_2)(z_3-z)=(w_1-w_2)(w_3-w)(z-z_1)(z_2-z_3)$$

Put $z=z_1$ then you get $w=w_1$

Put $z=z_2$ then you get $w=w_2$

Put $z=z_3$ then you get

$$(w-w_1)(w_2-w_3)=(w_3-w)(w_1-w_2)$$

which has unique solution $w = w_3$. Thus you see that expression (i) gives a transformation which maps z_1, z_2 and z_3 in the z-plane onto w_1, w_2 and w_3 in the w-plane respectively. Further you can check that equation (i) can be expressed in the form

$$w = \frac{Az + B}{Cz + D}$$

where A, B, C, D are constants involving z_1, z_2, z_3, w_1, w_2, w_3 such that $AD - BC \neq 0$

Hence (i) gives a bilinear transformation

Next we show that such a bilinear transformation is unique.

If possible, let T and S be two bilinear transformations which maps z_1, z_2 and z_3, onto w_1, w_2 and w_3 respectively. Then

$$Tz_1 = w_1,\ Tz_2 = w_2,\ Tz_3 = w_3$$

and $\quad Sz_1 = w_1,\ Sz_2 = w_2,\ Sz_3 = w_3$

As S s bilinear S^{-1} is bilinear and hence S^{-1} T is bilinear (being the composition of two bilinear maps)

We have

$$\left(S^{-1}T\right)z_1 = S^{-1}\left(Tz_1\right) = S^{-1}w_1 = z_1$$

Similarly $\left(S^{-1}T\right)z_2 = z_2$ and $\left(S^{-1}T\right)z_3 = z_3$

which shows that S^{-1} T has three fixed points But a bilinear tansformation can have move that two fixed points only if it is the identity map Hence

$$S^{-1}\, T = I$$

$$\Rightarrow S = T$$

6.3.4 Mapping of the upper half plane

We will find all bilinear transformation that map the upper half plane $Imz > 0$ onto the open disk $|w| < 1$ and the boundary $Imz = 0$ onto the boundary $|w| = 1$

Let the bilinear transformation be

$$w = Tz = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0 \text{ .... (i)}$$

$Imz > 0$ is to be mapped on to $|w| = 1$. Consider points on the line $Imz = 0$, say $z = 0, 1, \propto$. For these points we need to have $|w| = 1$.

When $z = 0$, $|w| = |b/d|$ from (i)

$$|w| = 1 \Rightarrow |b| = |d| \neq 0 \text{ ....(ii)}$$

Again we know for $z = \propto$, $w = a/c$ if $c \neq 0$

Hence $|w| = 1 \Rightarrow |a| = |c| \neq 0$ .... (iii)

Using (ii) and (iii) we can write

$$w = \frac{a}{c}.\frac{z + b/a}{z + d/c} \text{ .... (iv)}$$

Also from (ii) and (iii) we have $|b/a| = |d/c| \neq 0$ and $|a/c| = 1$

Let $b/a = -z_0$, $d/c = -z_1$, $a/c = e^{i\alpha}$,

where z_0, z_1 are non-zero complex constants and α is a real constant.

So, (iv) can be written in the form

$$w = e^{i\alpha}\frac{z - z_0}{z - z_1} \text{ .... (v)}$$

Again for $z=1$, $|w|=1$ gives

$$\left|\frac{1-z_0}{1-z_1}\right| = 1$$

i.e. $|1-z_1|=|1-z_0|$

$\Rightarrow |1-z_1|^2=|1-z_0|^2$

$\Rightarrow (1-z_1)\overline{(1-z_1)}=(1-z_0)\overline{(1-z_0)}$

$\Rightarrow (1-z_1)(1-\bar{z}_1)=(1-z_0)(1-\bar{z}_0)$

On simplifying and using $z_1\bar{z}_1=z_0\bar{z}_0$ $(\because |z_0|=|z_1|)$
we have

$z_1+\bar{z}_1=z_0+\bar{z}_0$

$\Rightarrow \operatorname{Re} z_1=\operatorname{Re} z_0$

We have $|z_0|=|z_1|$ and $\operatorname{Re} z_0=\operatorname{Re} z_1$, so either $z_0=z_1$ or $z_1=\bar{z}_0$

But $z_0=z_1$ will reduce (v) to a constant transformation. Hence

$z_1=\bar{z}_0$

So, the transformation is of the form

$$w=e^{i\alpha}\frac{z-z_0}{z-\bar{z}_0}$$

It shows that the point $z=z_0$ is mapped onto the origin of the w-plane. Again since points interior to the circle $|w|=1$ should be the image of points of the upper half plane we need to have $Imz_0>0$

Consequently, the bilinear transformation will have the form

$$w=e^{i\alpha}\frac{z-z_0}{z-\bar{z}_0} \quad Imz_0>0 \text{ .... (vi)}$$

where α is a real constent and z_0 a complex constent

Further we verify that transformation (vi) satisfy the stated conditions

Consider

$$|w|=\left|\frac{z-z_0}{z-\bar{z}_0}\right|$$

If a point z lie above the real axis both it and the point z_0 lie on the same side of that axis, which is the perpendicular bisector of the line segment joining z_0 and $\bar{z}_0$. It follows that the distance $|z-z_0|$ is less than the distance $|z-\bar{z}_0|$, that is $|w|<1$. Similarly if z lies below the real axis, the distance $|z-z_0|$ is greater than the distance $|z-\bar{z}_0|$, and so $|w|>1$. Finally if z is on the real axis, then $|z-z_0|=|z-\bar{z}_0|$ and hence $|w|=1$. Since any bilinear transformation is a one to one mapping of the extended z plane onto the extended w-plane, we conclude that the transformation (vi) maps the half-plane $\operatorname{Im} z>0$ onto the disk $|w|<1$ and the boundary of the half plane onto the boundary of the disk.

Remark : Let us see what happens when we replace the condition $\operatorname{Im} z_0>0$ by $\operatorname{Im} z_0<0$ in (vi)

We have

$$|w|<1 \Leftrightarrow |z-z_0|<|z-\overline{z_0}|$$

$$\Leftrightarrow |z-z_0|^2<|z-\overline{z_0}|^2$$

$$\Leftrightarrow (z-z_0)\overline{(z-z_0)}<(z-\overline{z_0})\overline{(z-\overline{z_0})}$$

$$\Leftrightarrow |z|^2+|z_0|^2-z\bar{z}_0-\bar{z}z_0<|z|^2+|z_0|^2-\overline{zz_0}-zz_0$$

$$\Leftrightarrow \operatorname{Re} z\bar{z}_0<\operatorname{Re} zz_0$$

$$\Leftrightarrow 2yy_0>0 \quad \text{Writing } x+iy=z$$

$$\Leftrightarrow y<0, \Leftrightarrow 2yy_0>0 \text{ and } x_0+iy_0=z_0$$

Thus the lower half plane is mapped onto $|w|<1$, and the boundary $y = 0$ is mapped onto $|w|=1$.

Example 1 : Consider the transformation

$$w = \frac{i-z}{i+z}$$

This can be written in the form

$$w = e^{i\pi}\frac{z-i}{z+i}$$

Comparing this with (vi), we get $\alpha = \pi$ and $Im z_0 = 1 > 0$ Hence this bilinear transformation maps $Im z_0 \geq 0$ onto $|w| \leq 1$.

Example 2 : Find all bilinear transformation which map the right half plane $Re(z) > 0$ onto the open disk $|w| < 1$ and the boundary $Re(z) = 0$ onto the boundary $|w|=1$.

Solution : We know that if we rotate the right half plane $Re(z) > 0$ and the real axis $\operatorname{Im} z = 0$, by an angle of $\pi/2$, we get the upper half plane $Imz > 0$ and the imaginary axis $Re(z) = 0$ respectively.

This transformation is given by

$$Gz = iz \qquad Re\ z \geq 0$$

Again the bilinear transformation which map $Imz \geq 0$ onto $|w| \leq 1$ is given by

$$Tz = e^{i\alpha}\frac{z-z_0}{z-\bar{z}_0}, \qquad \alpha \in \mathbf{R},\ Im\ z_0 > 0$$

Hence the required transformation is $(TG)z$

i.e. $(TG)z = T(Gz) = T(iz)$

$$= e^{i\alpha}\frac{iz - z_0}{iz - \bar{z}_0}, \ \alpha \in \mathbf{R}, \ Im\ z_0 > 0$$

Example 3 : Show that $w = \dfrac{z+i}{iz+I}$ maps $Im\ z \le 0$ onto $|w| \le 1$.

Solution : We have

$$w = \frac{z+i}{iz+I}$$

$$= \frac{z+i}{i(z-i)}$$

$$= -i\frac{z+i}{z-i}$$

$$= e^{-i\pi/2}\frac{z-(-i)}{z-(\overline{-i})}$$

Comparing this with (vi) we see that $\alpha = -\pi/2$ and $z_0 = -i$, i.e. $Im\ z_0 < 0$

Hence from the remark we conclude that the transformation map $Im\ z \le 0$ onto $|w| \le 1$

> **Check your progress**
>
> (g) Find the bilinear transformation that maps $\mathrm{Re}(z)<0$ onto $|w|<1$, and the boundary $\mathrm{Re}\, z = 0$ onto the boundary $|w| = 1$.

6.4 Conformal Mapping

In this section you will be introduced to the concept of conformal mappings. These are mapping which preserve angle between curves. First you will be aquanted with the notion of angle of rotation of curves under a transformation.

Let C be a smooth curve given by $z(t) = x(t) + iy(t), \ a \le t \le b$. So $z'(t)$ is continuous and non-zero $\forall t \in (a,b)$. We know that increasing values of the parametic t will give positive direction of the curve C. Let $f(z)$ be a function defined on C. Then

$$w(t) = f(z(t)), \qquad a \le t \le b$$

is the parametric representation of the image curve Γ of C under the transformation $w = f(z)$. Let C pass through some point z_0 . Suppose *f* is analytic at z_0 and $f(z_0) \neq 0$. Then by the chain rule

$$w'(t_0) = f(z(t_0))z'(t_0), \qquad a \le t_0 \le b$$

As $f'(z(t_0)) \neq 0$ we have

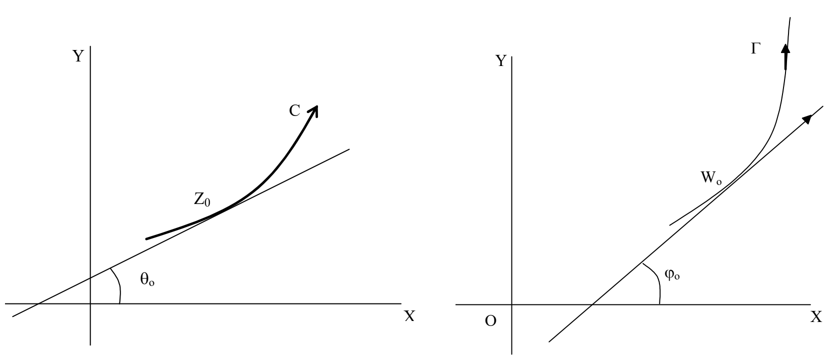
$$\arg w'(t_0) = \arg f'(z(t_0)) + \arg z'(t_0)$$

$$\Rightarrow \phi_0 = \psi_0 + \theta_0$$

where θ_0 is the angle of inclination of the directed tangent to C at z_0 and is a value of $\arg z'(t_0)$. Also ϕ_0 is the angle of inclination of the directed tangent to Γ (image curve), whereas ψ_0 is the value of $\arg f'(z_0)$

Hence the angle of inclination of the directed tangent at w_0 differs from the angle of inclination of the directed tangent at z_0 by the angle which we have denoted by $\psi_0 = \arg f'(z_0)$

This angle $\arg f'(z_0)$ is called the angle of rotation at z_0 .



Example 1 : Find the angle of rotation of a curve C under the transformation $f(z) = z^2 + 2i$ at the point 1-i.

Solution : The angle of rotation at $z_0 = 1 - i$ is given by $\arg f'(z_0)$

We have

$$f'(z_0) = [2z]_{1-i} = 2(1-i) = 2e^{-i\pi/4}$$

$\therefore$ Angle of rotation $= \arg f'(z_0) = -\pi/4$

> **Check your progress**
>
> (h) Find the angle of rotation of a curve C under the transformation $f(z) = Sin z$ at
>
> (a) i, (b) -i, (c) 1, (d) -1

Defintion **:** A mapping $f(z)$ is said to be conformal; at a point z_0 if it preserve the angle between oriented curves passing through z_0 in magnitude as well as in sense.

Geometrically this means that the images of any two oriented curves taken with their corresponding orientations make the same angle of intersection as the curves at z_0, both in magnitude and direction.

If $f(z)$ is conformal at all points of its domain then it is called a conformal mapping.

The following theorem gives the sufficient condition for a function to be conformal.

Theorem 1 **:** A mapping $f(z)$ is conformal at a point z_0 if it is analytic at z_0 and $f(z_0) \neq 0$.

Proof : Let C_1 and C_2 be two oriented curves through z_0 such that θ_1 and θ_2 are respectively the angles made by the directive tangents to the two curves at z_0. Let ϕ_1 and ϕ_2 be the angles made by the directed tangents to the image curves at w_0. Then from the previous discussion we know that the angle of rotation of the curves C_1 and C_2 at z_0 is given by

$$\arg f'(z_0) = \phi_1 - \theta_1 \text{ and } \phi_2 - \theta_2 = \arg f'(z_0)$$

Hence

$$\phi_1 - \theta_1 = \phi_2 - \theta_2$$

$$\Rightarrow \phi_2 - \phi_1 = \theta_2 - \theta_1$$

Thus the angle $\phi_2 - \phi_1$ from Γ_1 to Γ_2 is the same in magnitude and sense as the angle $\theta_2 - \theta_1$ from C_1 and C_2. Consequently f is conformal at z_0.

Remark : To verify that a function $f(z)$ is conformal at some point z_0 examine whether f is analytic at z_0 and $f'(z_0) \neq 0$.

Check your progress :

(i) Show that $f(z) = z^2 + iz$ is conformal at $z=i$ but not conformal at $z = -i/2$.

Example : Consider the transformation $w = f(z) = \bar{z}$. Under this transformation any curve or region get reflected along the real axis. So under their transformation angle between two curves is preserved in megnitude but not in sense.

These type of transformations which preserve magnitude but not sense are called isogonal mapping.

Exercise : The bilinear transformation

$$w = Tz = \frac{az+b}{cz+d}, \qquad ad - bc \neq 0$$

is conformal at all points except one.

Proof :

We have

$$Tz = \frac{az+b}{cz+d}$$

$$\therefore T'(z) = \frac{(cz+d)a-(az+b)c}{(cz+d)^2}$$

$$= \frac{ad-bc}{(cz+d)^2} \neq 0 \text{, as } ad-bc \neq 0$$

So Tz is analytic and $T'(z) \neq 0$ at all points except $z = -d/c$

Consequently Tz is conformal everywhere except at $z = -d/c$.

Definition **:** If $f(z)$ be analytic at z_0 and $f'(z_0) = 0$, then z_0 is called the critical point of the transformation.

Example : Consider the transformation

$$w = f(z) = e^{2z} - 2iz + z$$

$f(z)$ is non constant and analytic everywhere, the critical points are given by

$$f'(z) = 0 \text{, i.e. } 2e^{2z} - 2i = 0$$

$$\Rightarrow e^{2z} = i$$

$$\Rightarrow 2z = i\left(2n\pi + \pi/2\right),\ n \in I$$

i.e. $z = i\left(n\pi + \pi/4\right),\ n \in I$.

Definition **:** Let $f(z)$ be analytic at z_0 then the number $|f'(z_0)|$ is called the scalar factor of the mapping at z_0.

We have

$$|f'(z_0)| = \left| \underset{z\to z_0}{Lt} \frac{f(z)-f(z_0)}{z-z_0} \right| = \underset{z\to z_0}{Lt} \frac{|f(z)-f(z_0)|}{|z-z_0|}$$

$|z - z_0|$ is the length of the line segment joining z_0 and z and $|f(z) - f(z_0)|$ is the length of the line segment joining $f(z_0)$ and $f(z)$ in the w-plane

Remark : If $w = f(z)$ is conformal at z_0, then for points near z_0 ,the angle of rotation and scalar factor are approximately equal to *arg* $f'(z_0)$ and $|f'(z_0)|$ respectively. This mean that the image of a small region in a neighbourhood of z_0 conforms to the original region in the sanse that it has approximately the same shape.

> **Check your progress :**
>
> (j) Find the scalar factor of the transformation $w = z^2$ at $z = 2 + i$

Theorem **:** Let $f(z)$ be analytic at z_0. If f has a zero of order (*k-1*) at z_0 , then the mapping $w = f(z)$ magnifies the angle at z_0 by the factor k.

In other word , if α is the angle beftween two curves C_1 and C_2 in the z -plane, then the angle between the image curves Γ_1 and Γ_2 in the w- plane is $k\alpha$.

Proof : Since $f(z)$ is analytic at z_0, by Taylor servies

$$f(z) = \sum_0^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n$$

in some neighbourhood of z_0.

As $f'(z)$ has a zero of order (*k* - *1*) at z_0

$$f'(z_0) = \ldots\ldots\ldots\ldots\ldots = f^{k-1}(z_0) = 0 \text{ and } f^{(k)}(z_0) \neq 0$$

$\therefore f(z)$ has the representation

$$f(z) = f(z_0) + \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k + \frac{f^{(k+1)}(z_0)}{(k+1)!}(z-z_0)^{k+1} + \ldots$$

$$\Rightarrow f(z) - f(z_0) = (z-z_0)^k \left[\frac{f^{(k)}(z_0)}{k!} + \frac{f^{(k+1)}(z_0)}{(k+1)!}(z-z_0) + \ldots \right]$$

$$= (z-z_0)^k g(z) \text{ .... (i)}$$

where $g(z)$ is analytic at z_0 and

$$g(z_0) = \frac{f^{(k)}(z_0)}{k!} \neq 0$$

Then writing $w = f(z)$ and $w_0 = f(z_0)$ we have

$$\arg(w - w_0) = \arg\left[f(z) - f(z_0)\right] = k \arg(z - z_0) + \arg g(z) \text{ ...... (ii)}$$

Let C_0 be a smooth curve passing through z_0 and Γ be the image under $w = f(z)$.

Let $z \to z_0$ along C_0 then $w \to w_0$ along Γ. Also then the angle of inclination of tangents to C_0 and T_0 are given by

$$\theta_0 = \underset{z \to z_0}{Lt} \arg(z - z_0) \text{ and } \phi_0 = \underset{w \to w_0}{Lt} \arg(w - w_0)$$

Then from (ii) we have

$$\phi = k\theta + \arg g(z_0) \text{ .... (iii)}$$

Next C_1 and C_2 be two curves passing through z_0 and Γ_1 and Γ_2 be their images under $w = f(z)$. Then from (iii)

$$\phi_1 = k\theta_1 + \arg g(z_0)$$

and $\quad \phi_2 = k\theta_2 + \arg g(z_0)$

Where θ_1, ϕ_1 are associated to C_1 and T_1,

and θ_2, ϕ_2 are associated to C_2 and T_2.

Hence we have $\phi_2 - \phi_1 = k(\theta_2 - \theta_1)$

That is the angle from Γ_1 to Γ_2 is k- times the angle from C_1 and C_2 in magnitude. Further note that the direction is preserved

6.5 1. Let us sum up :

In this unit we have discussed different types of transformations.

Starting from elementry transformation like translation, rotation, stretching (contraction), linear, we have considered complex transformation like bilinear transformation. We have seen that bilinear transformation maps circle and line into circle s and lines. different properties of bilinear transformation were discussed. Some of them are : the inverse of a bilinear transformation has at most two fixed points, a bilinear transformations continuous in the extended complx plane, a bilinear transformation is analytic everywhere except possibly at a single point. One important result is that any three distinct in the z-plane can be mapped onto three specified points of the w-plane by a unique bilinear transformation. The bilinear transformation which maps the upper half plane Im z>0 onto the open disk $|w|$<0 and the boundary Imz=0 onto the boundary |w|=1 is of the form.

$$W = e^{i\alpha} \; (z - z_0)\left(z - -z\,\overline{z_0}\right)$$

2. Keywords :

Translation, Rotation, Contraction, Bilinear transformation, Fixed Points, Upper half plane, Lower half plane, Conformal mapping.

3. References :

1. R.V. Chufrchill & J.W. Brown, Complex Variables and Applications, Mc. Grow Hill
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4. Possible Answers to the CYP :

(a) The region is the interior and boundary of the circle $|z|=1$. The transformation is a translation. The region will be displaced through a distance of $\sqrt{5}$ along the vector (2,1)

(b) The Transformation is rotation through an angle of $\pi/4$.

(c) The Region $1 \le |z| \le 2$ is the annular region between the concentric circle $|z|=1$ and $|z|=2$ and the transformation $w=(1+i)z$ is a strecthing (since here $a=1+i$ and $|a|>1$)

(d) The given region is the circle $|z|=2$ and the transformation $w=iz+1$ is the combination of relation by 90°/ve multiplication by 9i) followed by a translation.

(e) Proceed as in the example 2 of that section.

(f) Proceed as in example 1 of that section

(g) The required transformation can be obtained by simply applying the transformation Hz=-z to the transformation obtained in example 2 of that section.

(h) Proceed as in the given example 1 of that section.

(i) $f(g) = z^2 + iz, \quad f^{/}(z) = 2z + i$

$$f^{/}(z) = 0 \quad for \quad z = {-i}/{2}$$

5. Model Questions :

1. Find the image of the infinite strip $0<x<1$, under the transformation $w = iz$.
2. Show that the transformation $w = iz + 1$ maps the half plane $x>0$ onto the half plane $v>0$.
3. Find the region onto which the half plane $y>0$ is mapped by the transformation $w = (1+i)\, z$
4. Find the image of the quadrant $x>1$, $y>0$ under the transformation $w = 1/z$.
5. Describe geometrically the transformation $w = \dfrac{1}{z-1}$
6. Find the bilinear transformation that maps the points $z_1 = 2$, $z_2 = i$, $z_3 = -2$ onto the points $w_1 = 1$, $w_2 = i$, $w_3 = -1$.
7. Find the bilinear transformation that maps the points $z_1 = -i, z_2 = 0, z_3 = i$ onto the points $w_1 = -1, w_2 = i, w_3 = 1$. Into what curve is the imaginary axis x=0 transfromed?
8. Find the bilinear tansformation that maps distinct points z_1, z_2, z_3 onto the points $w_1 = 0, w_2 = 1, w_3 = \propto$
9. Find the fixed point of the transformation

   (a) $w = \dfrac{z-1}{z+1}$, (b) $\dfrac{6z-9}{z}$

10. Prove that if the origin is a fixed point of a bilinear transformation. then the transformation is of the form

$$w = \frac{z}{cz + d}, \qquad d \neq 0$$

11. Determine the angle of rotation at the point $z = 2 + i$ when the transformation is $w = z^2$. Find the scalar factor of the transformation at that point.

12. What angle of rotation is produced by the transformation $w = 1/z$ at the point

(a) $z=1$, (b) $z=i$, (c) z = -1

13. Show that the transformation w = sin z is conformal at all points except

$$z = \pi/2 + n\pi \quad (n = 0, \pm1, \pm2, \ldots\ldots\ldots\ldots)$$

14. Construct a linear transformation which carries i onto -i and maps $1+2i$ onto itself.

15. Find the image of the strip $x>0$, and $0<y<1$ under the, map $w = i/z$

16. Determine all bilinear transformation which have fixed points as -i and i.

17. Find the bilinear transformation with fixed points-1 and 1 carrying the points i onto -i

18. Let $Tz = \dfrac{az+b}{cz+d}$, where $ad - bc \neq 0$ be any bilinear transformation other than Tz=z.

Show that $T^{-1} = T$ iff $d = -a$.

19. By finding the inverse of the transformation

$$w = \frac{i - z}{i + z}$$

Show that $w = i\frac{1 - z}{1 + z}$

maps the disk $|z| \leq 1$. onto the half plane $\text{Imw} \geq 0$

20. Let $w = \frac{z - i}{iz + 1}$. then show that

$\text{Im}\, z \leq 0 \Rightarrow |w| \leq 1$

21. Construct the general bilinear transformation which maps the upper half plane

(a) Onto itself, (b) Onto the lower half plane

22. Find the bilinear transformation which maps the upper half plane of the z-plane onto the unit circle in the w-plane in such a way that z = i is mapped onto w = 0 while the point at infinity is mapped onto w= -1.