SELF LEARNING MATERIAL

Master of Arts/Science

# MATHEMATICS

(2nd Semester)

COURSE: MATH - 203

# DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

BLOCK - 1, 2, 3, 4 & 5

Centre For Distance and Online Education
DIBRUGARH UNIVERSITY
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# MATHEMATICS

**COURSE : MATH - 203**

# DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

## BLOCK - 1, 2, 3, 4 & 5


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# MATHEMATICS

**COURSE : MATH - 203**

# DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

# BLOCK 1
# ORDINARY DIFFERENTIAL EQUATIONS

**Structure**



## 1.0 Introduction

Many problems in science, engineering and technology reduce to mathematical problems consisting of differential equations fall basically into two classes, ordinary and partial, depending upon the number of independent variables presents. Solutions of many differential equations arising out of physical problems are subject to some conditions.

In this unit we will introduce the concept of initial and boundary value problems. Questions may arise: Whether solution exists for all problems? Under what conditions does the problem has at least one solution? Under what conditions does the problem has a unique solution? These questions will also be answered in this unit.

Finally, dependence of solutions and the concept of Wronskian will also be introduced.

## 1.1 Objectives

After studying this block you should be able to:

- Know occurrence and origin of differential equations in various fields.
- Explain, what an initial value problem (IVP) and a boundary value problem (BVP) is
- Know the Sturm-Liouvielle problem

- Know the conditions for existence and uniqueness of solutions of an ordinary differential equations
- Understand the concept of dependence of solutions, Wronskin and its use in Linear independence of solutions.

## 1.2 Initial Value Problem

An equation of the form

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

where $y(x_0)$ denotes the value of y at $x=x_0$ is called initial value problem.

Initial value problem can be solved by two methods

(i) The Euler method

Let y denote the exact solution of the initial value problem which consists of differential equation

$$\frac{dy}{dx} = f(x, y) \qquad (1.2.1)$$

and the initial condition $y(x_0) = y_0$ (1.2.2)

Let h denote the positive increment in x and let

$x_1 = x_0 + h$ so that $x_1 - x_0 = h$ (1.2.3)

Now $\int_{x_0}^{x_1} f(x, y)dx = \int_{x_0}^{x_1} \frac{dy}{dx} dx = y(x_1) - y(x_0) = y(x_1) - y_0$

$$\therefore y(x_1) = y_0 + \int_{x_0}^{x_1} f(x, y)dx \qquad (1.2.4)$$

If we assume that f(x,y) varies slowly on the interval $x_0 \le x \le x_1$, then we can approximate f(x,y) in (1.2.4) by its value $f(x_0,y_0)$ at the left hand point $x_0$. Thus we have

$$y(x1) \approx y0 + \int_{x0}^{x1} f(x_0, y_0)dx \qquad (1.2.5)$$

where $\approx$ stands for approximation.

But $\int_{x_0}^{x_1} f(x_0, y_0)dx = f(x_0, y_0) \int_{x_0}^{x_1} dx = f(x_0, y_0)(x_1 - x_0)$

$$hf(x_0, y_0)$$

$\therefore$ (1.2.5) becomes $y(x_1) \approx y_0 + hf(x_0, y_0)$

It follows that the approximate value $y(x_1)$ i.e., $y_1$ of y at $x_1$ is given by the formula

$$y_1 = y_0 + hf(x_0, y_0) \qquad (1.2.6)$$

with the value of $y_1$ given by (1.2.6) we now proved in like manner to get $y_2$ by the formula

$$y_2 = y_1 + hf(x_1, y_1) \qquad (1.2.7)$$

In general by the repeated application of the above method, we determine $y_{n+1}$ in terms of $y_n$ by the formula

$$y_2 = y_1 + hf(x_1, y_1) \qquad (1.2.7)$$

In general by the repeated application of the above method, we determine $y_{n+1}$ in terms of $y_n$ by the formula

$$y_{n+1} = y_n + hf(x_n, y_n) \qquad (1.2.8)$$

(ii) Picard's method

Consider an initial value problem of the form

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \qquad (1.2.9)$$

By integrating over the interval $(x_0,x)$, (1.2.9) gives

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x, y)dx$$

or $$y(x) - y_0 = \int_{x_0}^{x} f(x, y)dx$$

or $$y(x) = y_0 + \int_{x_0}^{x} f(x, y)dx \qquad (1.2.10)$$

Thus, the solving of initial value problem (1.2.9) is equivalent to inding a function y(x) which satisfies the equation (1.2.10), since by differentiating (1.2.10) we get $\frac{dy}{dx} = f(x, y)$ and putting $x=x_0$ in (1.2.10) yields $y(x_0)=y_0+0$ i.e., $y(x_0)=y_0$. Conversly (1.2.10) has been obtained from (1.2.9) by integrating over the interval $(x_0,x)$ and employing the initial condition $y(x_0)=y_0$.

Since the information concerning the expression of y in term of x is absent, the integral on the RHS of (1.2.10) cannot be evaluated. Hence the exact value of y cannot be obtained. Therefore we determine a sequence of approximation, we put $y=y_0$ in the integral on the right of (1.2.10) and obtain.

$$y_1(x) = y_0 + \int_{x_0}^{x_1} f(x, y_0)dx \qquad (1.2.11)$$

where $y_1(x)$ is the corresponding value of y(x) and is called first approximation and is better approximation of y(x) at any x. To determine still better approximation we replace y by $y_1$ in the integral on RHS in (1.2.10) and obtain the second approximation $y_2$ as

$$y_2(x) = y_0 + \int_{x_0}^{x_1} f(x, y_1)dx \quad (1.2.12)$$

Proceeding in this way, the nth approximation $y_n$ is given by

$$y_n(x) = y_0 + \int_{x_0}^{x_1} f(x, y_{n-1})dx \quad (1.2.13)$$

Thus, we arrive at a sequence of approximate solution
$y_1(x), y_2(x), \ldots, y_n(x)$

## 1.3 Sturm – Liouvielle Problem

A special kind of boundary value problems are known as Strum-Lioville problems. Such problems arise in physics and engineering and help solving boundary value problems of partial differential equations.

Definition. A second order differential of the form

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0$$

where pm q and r are real functions, p has a continuous derivative, q and r are continuous; p(x)>0, r(x)>0 for all $x \in [a,b]$, $\lambda$ is a parameter independent of x and y satisfies the boundary conditions

$$A_1 y(a) + A_2 y'(a) = 0, B_1 y(a) + B_2 y'(b) = 0$$

A1, A2, B1, B2 being real constants such that A1 and A2 are not both zero, B1 and B2 are not both zero, is called a Strum-Liouville problem.

Example : The boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, y(0) = 0, y(\pi) = 0$$

is a Sturm-Liouville problem, as the equation can be written as

$$\frac{d}{dx}\left[1.\frac{dy}{dx}\right] + [0 + \lambda.1]y = 0$$

with the boundary conditions

$$1y(0) + 0y'(0) = 0$$
$$1y(\pi) + 0y'(\pi) = 0$$

**Definition**: The values of the parameter $\lambda$ of a Sturm-Lioville problem for which there exist non-trivial solutions of the problem are called eigen values or characteristic roots of the problem. The corresponding non-trivial solutions themselves are called eigen functions or characteristic functions of the problem.

## 1.4 Existence and Uniqueness of Solution

Just as a differential equation of the first order need not have a solution always, an equation of the second or higher order may not have a solution. However, a set of sufficient conditions has been devised a guarantee the existence and uniqueness of a solution of such an equation. Since the theorem of existence and uniqueness for an nth order equation may readily be derived by reducing I to a system of equations for which the existence and uniqueness has already been proved, we state here the related conditions and don't give the proof of the theorem.

### 1.3.1 Existence and Uniqueness theorems

There exists a unique solution of an nth order differential equation $y^{(n)} = f(x, y', ..., y^{(n-1)})$ that satisfies the conditions

$$y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0, ... y^{(n-1)}(x_0) = y_0^{(n-1)}$$

is the neighbourhood of the initial values ($x_0$, $y_0$, $y'_0$,...,$y^{(n-1)}_0$), the function f is a continuous function of all of the initial arguments and satisfies the Lipschitz condition with respect to all arguments from the second onwards.

**Statement**. Let $f(x,y)$ be continuous in a domain D of the $(x,y)$ plane and let M be a constant such that (1.4.1.1)

$|f(x,y)| \le M$ ......... in D. Let $f(x,y)$ satisfy in D the Lipschitz condition in $y$ namely

$$|f(x,y_1) - f(x,y_2)| \le K|y_1 - y_2| \quad ... (1.4.1.2)$$

where constant K is independent of $x, y_1, y_2$.

Let the rectangle R, defined by

$$|x - x_0| \le h, |y - y_0| \le k, \quad ... (1.4.1.3)$$

lie in D, where $Mh < k$. Then ,for $|x - x_0| \le h$, the differential equation $dy/dx = f(x,y)$ has a unique solution $y = y(x)$ for which $y(x_0)=y_0$

**Proof:**

We shall prove this theorem by the method of successive approximations. Let x be such that $|x-x_0| \le h$. We now define a sequence of functions $y_1(x), y_2(x), \ldots y_n(x)\ldots$, called the successive approximations(or Picard Iterants) as follows:

$$\left.\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(x, y_0)dx \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(x, y_1)dx \\ \ldots \quad & \ldots \quad \ldots \quad \ldots \\ y_{n-1}(x) &= y_0 + \int_{x_0}^{x} f(x, y_{n-2})dx \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx \end{aligned}\right\} \quad \ldots (1.4.1.4)$$

We shall divide the proof into five main steps.

**First Step:**

We prove that, for $x_0 - h \le x \le x_0 + h$, the curve $y = y_n(x)$ lies in the rectangle R, that is to say $y_0 - k < y < y_0 + k$.

Now $|y_1 - y_0| = \left| \int_{x_0}^{x} f(x, y_0)dx \right|$

$$\le \int_{x_0}^{x} |f(x, y_0)|.|dx|, \text{ by} \quad \ldots (1.4.1.4)$$

or $|y_1 - y_0| \le M|x - x_0| \le Mh < k,$

By (1.4.1.1),(1.4.1.3) and the given result viz. $Mh < k$.

This proves the desired result for n=1.Assume that

$$y = y_{n-1}(x)$$

lies in R and so $f[x, y_{n-1}]$ is defined and continuous and satisfies

$$|f(x, y_{n-1}| \le M \text{ on } [x_0 - h, x_0 + h].$$

From (4),we have $|y_n - y_0| = \left| \int_{x_0}^{x} f(x, y_{n-1})dx \right|$

$$\le \int_{x_0}^{x} |f(x, y_{n-1})|.|dx|$$

$$\leq M|x - x_0| \leq Mh < k,$$

as before which shows that $y_n(x)$ lies in R and hence $f(x, y_n)$ is defined and continuous on $[x_0 - h, x_0 + h]$.The above arguments show that the desired result holds for all n by induction.

**Second Step.**

We prove again by induction, that

$$|y_n - y_{n-1}| \leq \frac{MK^{n-1}}{n!}|x - x_0|^n \qquad \ldots (1.4.1.5)$$

We have already verified (1.4.1.5) for n=1 in first step where we have shown that $|y_1 - y_0| \leq M|x - x_0|$.Assume that this inequality (1.4.1.5) holds for n-1 in place of n, that is let

$$|y_{n-1} - y_{n-2}| \leq \frac{MK^{n-2}}{(n-1)!}|x - x_0|^{n-1} \qquad \ldots (1.4.1.6)$$

Then , we have

$$|y_n - y_{n-1}| = \left|\int_{x_0}^{x}\{f(x, y_{n-1}) - f(x, y_{n-2})\}dx\right| \text{ by } \qquad (1.4.1.4)$$

or $$|y_n - y_{n-1}| \leq \int_{x_0}^{x}|f(x, y_{n-1}) - f(x, y_{n-2})|dx \qquad (1.4.1.7)$$

Lipschitz condition (1.4.1.2) gives

$$|f(x, y_{n-1}) - f(x, y_{n-2})| \leq K|y_{n-1} - y_{n-2}| \quad \ldots (1.4.1.8)$$

Form (7) and (8) ,we get

$$|y_n - y_{n-1}| \leq \int_{x_0}^{x} K|y_{n-1} - y_{-2}|.|dx|$$

$$\leq K.\frac{MK^{n-1}}{(n-1)!}.\frac{|x - x_0|^n}{n}, \qquad \text{by } (1.4.1.6)$$

Hence by mathematical induction, we conclude that (1.4.1.5) is true for each natural number n.

**Third Step.**

We shall now prove that the sequence $y_n$ converges uniformly to a limit for $x_0 - h \leq x \leq x_0 + h$.

For the interval under consideration, $|x - x_0| \le h$.
Hence from second step, we have

$$|y_n - y_{n-1}| \le \frac{MK^{n-1}h^n}{n!} \text{ is true for all n.}$$

Using this ,the infinite series

$$y_0 + (y_1 - y_0) + (y_2 - y_1) + ... + (y_n - y_{n-1}) + ... \quad ...(1.4.1.9)$$
$$\le y_0 + Mh + \frac{1}{2!}MKh^2 + ... + \frac{1}{n!}MK^{n-1}h^n + ...$$
$$\le y_0 + \frac{M}{K}\left[e^{Kh} - 1\right]$$

which is known to be convergent for all values of K,h and M. Consequently, the series (1.4.1.9) is surely convergent. Thus, by the Weirstrass M-test, the series (1.4.1.9) converges uniformly on $\left[x_0 - h, x_0 + h\right]$.Now since the terms of (1.4.1.9) are continuous function of x, its sum

$$\lim_{n\to\infty} y_n(x) = y(x), \text{ say,} \quad ... (1.4.1.10)$$
$$\left[\because y_n = y_0 + \sum_{n=1}^{n}(y_n - y_{n-1})\right]$$

must be continuous.

**Fourth Step.**

We now show that $y = y(x)$ satisfies the differential equation $dy/dx = f(x, y)$.

Since $y_n(x)$ tends uniformly to $y(x)$ in $\left[x_0 - h, x_0 + h\right]$ and by Lipschitz condition,

$$|f(x, y) - f(x, y_n)| \le K|y - y_n|,$$

it follows that $f\left[x, y_n(x)\right]$ tends uniformly to $f\left[x, y(x)\right]$. Again from (1.4.1.4) we have

$$y_n(x) = y_0 + \int_{x_0}^{x} f\left[x, y_{n-1}(x)\right]dx$$

or $\lim_{n\to\infty} y_n(x) = y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f\left[x, y_{n-1}(x)\right]dx$, ,letting $n \to \infty$.

Since the sequence $f[x, y_n(x)]$, consisting of continuous functions on the given interval, converge uniformly to $f[x, y(x)]$ on the same interval, the interchanges of limiting operations given bellow are valid .Thus using (1.4.1.10), we have

$$y(x) = y_0 + \int_{x_0}^{x} \lim_{n\to\infty} f[x, y_{n-1}(x)]dx$$

or $$y(x) = y_0 + \int_{x_0}^{x} f[x, y(x)]dx \qquad \ldots (1.4.1.11)$$

the integrand on the right-hand side of (1.4.1.11) being a continuous function of x, we conclude that the integral has the derivative thus, the limit function $y(x)$ satisfies the differential equation $dy/dx = f(x, y)$ on $[x_0 - h, x_0 + h]$ and is such that $y(x_0) = y_0$.

[In the above four steps we have thus proved the existence of a solution of the given initial value problem. The next step will show that the solution y(x) is unique.

**Fifth Step.**

**Uniqueness of the solution:**

We now prove that the solution $y = y(x)$ just found is the only solution for which $y(x_0) = y_0$.

Assume if possible $y = Y(x)$, say, is another solution of the given initial value problem.

Let $|Y(x) - y(x)| \le B$ when $x_0 - h \le x \le x_0 + h$ .. (1.4.1.12)

It may be noted here that we can surely take B=2K.

From (1.4.1.11), we get

$$|Y(x) - y(x)| = \left| \int_{x_0}^{x} [f\{x, Y(x)\} - f\{x, y(x)\}] \right|$$

or $$|Y(x) - y(x)| = \int_{x_0}^{x} |f\{x, Y(x)\} - f\{x, y(x)|.|dx|$$

or $$|Y(x) - y(x)| \le K \int_{x_0}^{x} |Y(x) - y(x)|.|dx| \qquad \ldots (1.4.1.13)$$

$[\because |f\{x, Y(x)\} - f\{x, y(x)| \le K|Y(x) - y(x)|$ by Lipschitzcondition$]$

or $$|Y(x) - y(x)| \le KB|x - x_0|, \text{using (12)} \qquad \ldots (1.4.1.14)$$

Now substituting (14) for integrand in (13) ,we get

$$|Y(x)-y(x)| \le K^2 B \int_{x_0}^{x} |x-x_0| dx \le \frac{K^2 B |x-x_0|^2}{2!} \qquad \dots (1.4.1.15)$$

again substituting (1.4.1.15) for integrand in (1.4.1.13) ,we get

$$|Y(x)-y(x)| \le \frac{K^3 B}{2!} \int_{x_0}^{x} |x-x_0|^2 dx \le \frac{K^3 B |x-x_0|^3}{3!}$$

Continuing in this way , we shall surely get

$$|Y(x)-y(x) \le \left| \frac{K^n B |x-x_0|^n}{n!} \le B\frac{(Kh)^n}{n!} \right. \qquad \dots (1.4.1.16)$$

Now the series $\sum_{n=0}^{n} B\frac{(Kh)^n}{n!}$ converges, and so $\lim_{n\to\infty} B\frac{(Kh)^n}{n!} = 0$.

Thus $|Y(x)-y(x)|$ can be made less than any number however small and consequently we conclude that

$$Y(x)-y(x)=0 \text{ i.e. } Y(x)=y(x).$$

This shows that the solution $y = y(x)$ is always unique, and the proof of the theorem is complete.

## 1.4 Linear dependence and independence of solutions of an equation

Let $y_1, y_2, \dots, y_n$ be n solutions of the linear differential equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + P_n y = 0$$

where $P_1, P_2, \dots P_n$ are functions of x or constant.

There n solutions $y_1, y_2, \dots, y_n$ are said to be linearly dependent, if there exists a set of n constants $a_1, a_2, \dots, a_n$ at least one of which is different from zero, such that

$$a_1 y_1 + a_2 y_2 + \dots\dots\dots + a_n y_n = 0$$

If no such set of n constants $a_1, a_2, \dots, a_n$ exists then the solutions $y_1, y_2, \dots, y_n$ are said to be linearly independent.

## 1.5 Wronskian

The determinant

$$W(x)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{n-1} & y_2^{n-1} & \cdots & y_n^{n-1} \end{vmatrix}$$

is called the Wronskian of the set of functions $y_1, y_2, y_3, \ldots, y_n$.

**Example.** Show that the Wronskian of $e^{ax}\cos bx$ and $e^{ax}\sin bx\ (b \neq 0)$ is $be^{2ax}$

Sol. Let $y_1 = e^{ax}\cos bx$ and $y_2 = e^{ax}\sin bx$

Then, Wronskian of $y_1, y_2$ is

$$W(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$=\begin{vmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ e^{ax}(a\cos bx - b\sin bx) & e^{ax}(a\sin bx + b\cos bx) \end{vmatrix}$$

$$= be^{2x}$$

## 1.7 Self assessment question.

1. Show that $f(x,y) = xy^2$ satisfies the Lipschitz condition on rectangle $|x| \le 1, |y| \le 1$ but does not satisfy a Lipschitz condition on the strip $|x| \le 1, |y| < \infty$.

2. Show that the conditions for the existence and uniqueness of a solution of the following initial value problem are not satisfied by the function $f(x,y) = (y-1)/x$ in any rectangle R of xy-plane with (0,1) as its centre : $y' = (y-1)/x, y(0) = 1$ but a solution does exist of above. Give reason for your answer. Draw some possible solution curves.

3. Show that the solutions $e^x, e^{-x}$ and $e^{2x}$ of $y_3 - 2y_2 - y_1 + 2y = 0$ are linearly independent and hence or otherwise solve the given equation.

## 1.8 Let us sum up:

In this block we have covered the following points:

1. We give the concept, how does a differential equation originates.
2. We introduced the concept of initial and boundary value problem and defined the Sturm-Liouvielle problem
3. We discussed the existence and uniqueness of solutions
4. We gave some theorem with proof on dependence of solutions
5. We introduced the Wronskian and its use in linear independence of solutions.

BLOCK 2
# PARTIAL DIFFERENTIAL EQUATIONS OF THE FIRST AND SECOND ORDER

**Structure**



## 2.0 Introduction

An equation involving partial differential co-efficient of a function of two or more independent variables is known as a partial differential equation. If a partial differential equation contains nth and lower order derivatives, it is said to be an nth 'order' equation. The 'degree' of such equation is the greatest exponent of the highest order. The equation will be called 'linear' if, it is of first degree in the dependent variable and its partial derivatives (i.e. the products or powers of the dependent variable and its partial derivative must be absent). An equation containing power or products of the dependent variable and its derivatives is called a 'non-linear' partial differential equation.

If $x$, $y$ are the independent variables and $z$ is the dependent variable, the following notations are used

$$p=\frac{\partial z}{\partial x},\quad q=\frac{\partial z}{\partial y},\quad r=\frac{\partial^2 z}{\partial x^2},\quad s=\frac{\partial^2 z}{\partial x\partial y},\quad t=\frac{\partial^2 z}{\partial y^2}$$

if $x_1, x_2, \ldots, x_n$, are the independent variables then

$$p_1=\frac{\partial z}{\partial x_1},\quad p_2=\frac{\partial z}{\partial x_2},\ldots,p_n\frac{\partial z}{\partial x_n}$$

are also used.

Sometimes, partial derivatives are denoted by making use of suffices. Thus

$$u_x=\frac{\partial u}{\partial x},\quad u_y=\frac{\partial u}{\partial y},\quad u_z=\frac{\partial u}{\partial z},\quad u_{xx}=\frac{\partial^2 u}{\partial x^2},\quad u_{xz}=\frac{\partial^2 u}{\partial x\partial z}\text{ etc.}$$

## 2.1 Objectives

After studying this unit, you will be able to:

- Know what a partial differential equation is, how does it occur and how does a PDE be obtained.
- Know what an integral surface and what an orthogonal surface is
- Solve equations using Charpit's and Jacobi's method
- Solve linear second order equations
- Reduce to canonical forms
- Solve non-linear equations using Monge's method.

## 2.2. Linear First order Partial Differential Equations

We now proceed to the study of partial differential equations proper. Such equations arise in geometry and physics when the number of independent variables in the problem under discussion is two or more. When such is the case, any dependent variable is likely to be a function of more than one variable, so that it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several variables.

For instance, in the study of thermal effects in a solid body the temperature θ many vary from point to point in the solid as well as from time to time, and, as consequence, the derivatives

$$\frac{\partial\theta}{\partial x},\ \frac{\partial\theta}{\partial y},\ \frac{\partial\theta}{\partial z},\ \frac{\partial\theta}{\partial t},$$

will, in genera, be nonzero. Furthermore in any particular problem it may happen that higher derivatives of the type

$$\frac{\partial^2\theta}{\partial x^2},\ \frac{\partial^2\theta}{\partial x\partial t},\ \frac{\partial^3\theta}{\partial x^2\partial t},\ \text{etc.}$$

may be of physical significance.

When the laws of physics are applied to a problem of this kind, we sometimes obtains a relation between the derivatives of the kind

$$F(\frac{\partial\theta}{\partial x},\ldots\ldots,\frac{\partial^2\theta}{\partial x^2},\ldots\ldots,\frac{\partial^2\theta}{\partial x\partial t},\ldots\ldots)=0 \qquad (2.2.1)$$

Such an equation relating partial derivatives is called a *partial differential equation*.

Just as in the case of ordinary differential equations, we define the order of a partial differential equation to be the order or the derivative of highest order occurring in the equation. If, for example, we take θ variables to be dependent variable and x, y and t to be independent variables, then the equation

$$\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t} \qquad (2.2.2)$$

is a second-order equation is two variables, the equation

$$\left(\frac{\partial\theta}{\partial x}\right)^3+\frac{\partial\theta}{\partial t}=0 \qquad (2.2.3)$$

is a first-order equation in two variables, while

$$x\frac{\partial\theta}{\partial x}+y\frac{\partial\theta}{\partial y}+\frac{\partial\theta}{\partial t}=0 \qquad (2.2.4)$$

is a first-order equation in three variables.

In this chapter we shall consider partial differential equations of the first order, i.e., equations of the type

$$F\left(\theta, \frac{\partial \theta}{\partial x}, \ldots\right) = 0 \qquad (2.2.5)$$

In the main we shall suppose that there are two independent variables x and y and that the dependent variable is denoted by z. If we write

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y} \qquad (2.2.6)$$

we see that such an equation can be written in the symbolic form

$$f(x, y, z, p, q) = 0 \qquad (2.2.7)$$

## 2.2.1. Origins of First-order Partial Equations

Before discussing the solutions of the type (2.2.7) of the last section, we shall examine the interesting question of how they arise. Suppose that we consider the equation

$$x^2 + y^2 + (z - c)^2 = a^2 \qquad (2.2.1.1)$$

in which the constants a and c are arbitrary. Then equation (2.2.1.1) represents the set of all spheres whose centers lie along the z axis. If we differentiate this equation with respect to x, we obtain the relation

$$x + p(z - c) = 0$$

while if we differentiate it with respect to y, we find that

$$y + q(z - c) = 0$$

Elimination the arbitrary constant c from these two equations, we obtain the partial differential equation

$$yp - xq = 0 \qquad (2.2.1.2)$$

which is of the first order. In some sense, then, the set of all spheres with centres on the z axis is characterized by the partial differential equation (2.2.1.2).

However, other geometrical entities can be described by the same equation. For example, the equation

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha \qquad (2.2.1.3)$$

in which both of the constants c and α are arbitrary, represents the set of all right circular cones whose axes coincide with the line Oz.

If we differentiate equation (2.2.1.3) first with respect to x and then with respect to y, we find that.

$$p(z-c)\tan^2\alpha = x, \qquad q(z-c)\tan^2\alpha = y \qquad (2.2.1.4)$$

and, upon eliminating c and α from these relations, we see that for these cones also the equation (2.2.1.2) is satisfied.

Now what the spheres and cones have in common is that they are surfaces of revolution which have the line Oz as axes of symmetry. All surfaces of revolution with this property are characterized by an equation of the form

$$z = f(x^2 + y^2) \qquad (2.2.1.5)$$

where the function $f$ is arbitrary. Now if we write $x^2 + y^2 = u$ and differentiate equation (2.2.1.5) with respect to x and y, respectively, we obtain the relations

$$p = 2xf'(u), \qquad q = 2yf'(u)$$

where $f'(u) = df/du$, from which we obtain equation (2) by eliminating the arbitrary function $f(u)$.

Thus we see that the function z defined by each of the equations (2.2.1.1), (2.2.1.3), and (2.2.1.5) is, in some sense, a "solution" of the equation (2.2.1.2).

We shall now generalize this argument slightly. The relations (2.2.1.1) and (2.2.1.3) are both of the type

$$F(x, y, z, a, b) = 0 \qquad (2.2.1.6)$$

where a and b denote arbitrary constants. If we differentiate this equation with respect to x, we obtain the relation

$$\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0, \qquad \frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \qquad (2.2.1.7)$$

The set of equations (2.2.1.6) and (2.2.1.7) constitute three equations involving two arbitrary constants a and b, and, in the general case, it will be possible to eliminate a and b from these equations to obtain a relation of the kind.

$$f(x, y, z, p, q) = 0 \qquad (2.2.1.8)$$

showing that the system of surfaces (2.2.1.1) gives rise to a partial differential equation (2.2.1.8) of the first order.

The obvious generalization of the relation (2.2.1.5) is a relation between x, y and z of the type

$$F(u, v) = 0 \qquad (2.2.1.9)$$

where u and v are known functions of x, y and z and F is an arbitrary function of u and v. If we differentiate equation (2.2.1.9) with respect to x and y, respectively, we obtain the equations

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right\}+\frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right\}=0$$

$$\frac{\partial F}{\partial u}\left\{\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right\}+\frac{\partial F}{\partial v}\left\{\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right\}=0$$

and if we now eliminate $\partial F/\partial u$ and $\partial F/\partial v$ from these equations, we obtain the equation

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \qquad (2.2.1.10)$$

which is a partial differential equation of the type (2.2.1.8).

It should be observed, however, that the partial differential equation (2.2.1.10) is a linear equation; i.e., the powers of p and q are both unity, whereas equation (2.2.1.8) need not be linear. For example, the equation

$$(x-a)^2+(y-b)^2+z^2=1$$

which represents the set of all spheres of unit radius with centre in the plane xOy, leads to first-order non-linear differential equation

$$z^2(1+p^2+q^2)=1$$

## 2.2.2. Cauchy's Problem for First-order Equations

Though a complete discussion of existence theorems would be out of place in a work of this kind, it is important that, even at this elementary stage, the student should realize just what is meant by an existence theorem. The business of a existence theorem is to establish conditions under which we can assert whether or not a given partial differential equation has a solution at all; the further step of proving that the solution, when it exists, is unique requires a uniqueness theorem. The conditions to be satisfied in the case of a first-order partial differential equation are conveniently crystallized in the classic problem of Cauchy, which in the case of two independent variables may be stated as follows:

*Cauchy's Problem.* If

(a) $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ are functions which, together with their first derivatives, are continuous in the interval M defined by $\mu_1 < \mu < \mu_2$;

(b) And if F(x, y, z, p, q) is a continuous function of x, y, z, p and q in a certain region U of the xyzpq space, then it is required to establish the existence of the functions $\phi(x, y)$ with the following properties:

(1) $\phi(x, y)$ and its partial derivatives with respect to x and y are continuous function of x and y in a region R of the xy space.

(2) For all values of x and y lying in R, the point $\{x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)\}$ lies in U and

$$F[x, y, \phi(x, y), \phi_x(x, y), \phi_y(x, y)] = 0$$

(3) For all m belonging to the interval M, the point $\{x_0(\mu), y_0(\mu)\}$ belong to the region R, and

$$\phi\{x_0(\mu), y_0(\mu)\} = z_0$$

Stated geometrically, what we wish to prove is that there exists a surface $z = \phi(x, y)$ which passes through the curve Γ whose parametric equations are

$$x = x_0(\mu), \quad y = y_0(\mu), \quad z = z_0(\mu), \qquad (2.2.2.1)$$

and at every point of which the direction $(p, q, -1)$ of the normal is such that

$$F(x, y, z, p, q) = 0 \qquad (2.2.2.2)$$

We have given only one form of the problem of Cauchy. The problem can in fact be formulated in seven other ways which are equivalent to the formulation above. The significant point is that the theorem can not be proved with this degree of generality. To prove the existence of the solution of equation (2.2.2.2) passing through a curve with equations (2.2.2.1) it is necessary to make some further assumption about the form of the function F and the nature of the curve Γ. There are, therefore, a whole class of existence theorems depending on the nature of the special assumptions. We shall not discuss these existence theorems here but shall content ourselves with quoting one of them to show the nature of such a theorem. For the proof of it the reader should consult pages 32 to 36 of Bernstein's monograph cited above. The classic theorem in this field is that due to Sonia Kowalewski:

**Theorem 1.** *If $g(y)$ and all its derivatives are continuous for $|y - y_0| < \delta$, if $x_0$ is a given number and $z_0 = g(y_0)$, $q_0 = g'(y_0)$, and if $f(x, y, z, q)$ and all its partial derivatives are continuous in a region S defined by*

$$|x - x_0| < \delta, \quad |y - y_0| < \delta, \quad |q - q_0| < \delta$$

*then there exists a unique function $\phi(x, y)$ such that:*

*(a) $\phi(x, y)$ and all its partial derivatives are continuous in a region R defined by*

$|x - x_0| < \delta_1, \quad |y - y_0| < \delta_2,$

*(b) For all (x, y) in R, $z = \phi(x, y)$ is solution of the equation*

$$\frac{\partial z}{\partial x} = f\left(x, y, z, \frac{\partial z}{\partial y}\right)$$

*(c) For all values of y in the interval $|y - y_0| < \delta_1$, $\phi(x_0, y) = g(y)$.*

Before passing on to the discussion of the solution of first-order partial differential equations, we shall say a word about different kinds of solutions. We saw in Sec. 2 that relations of the type

$$F(x, y, z, a, b) = 0 \qquad (2.2.2.3)$$

led to partial differential equations of the first order. Any such relation which contains two arbitrary constants a and b and is a solution of a partial differential equation of the first order is said to be a complete solution or a complete integral of the equation. On the other hand any relation of the type

$$F(u, v) = 0 \qquad (2.2.2.4)$$

involving an arbitrary function F connecting two known functions u and v fo x, y and z and providing a solution of a first-order partial differential equation is called a general solution or a general integral of that equation.

It is obvious that in some sense a general integral provides a much broader set of solutions of the partial differential equation in question than does a complete integral. We shall see later, however, that this is purely illusory in the sense that it is possible to derive a general integral of the equation once a complete integral is known.

## 2.2.3. Linear Equations of the First Order

We have already encountered linear equations of the first order in Sec. 2. They are partial differential equations of the form

$$Pp + Qq = R \qquad (2.2.3.1)$$

where P, Q and R are given functions of x, y and z (which do not involve p or q), p denotes $\partial z / \partial x$, $q$ denotes $\partial z / \partial y$, and we wish to find a relation between x, y and z involving an arbitrary function.

The first systematic theory of equations of this type was given by Lagrange. For that reason equation (2.2.3.1) is frequently referred to as Lagrange's equation. Its generalization to n independent variables is obviously the equation

$$X_1 p_1 + X_2 p_2 + \dots\dots\dots + X_n p_n = Y \qquad (2.2.3.2)$$

where $X_1$, $X_2$, …, $X_n$, and Y are functions of n independent variables $x_1$, $x_2$, …, $x_n$ and a dependent variable *f*; $p_i$ denotes $\delta f/\delta x_i$ (i = 1, 2, …, n). It should be observed that in this connection the team "linear" means that p and q (or, in general case, $p_1$, $p_2$, …, $p_n$) appear to the first degree only but P, Q, R may be any functions of x, y, and z. This is in contrast to the situation in the theory of ordinary differential equations, where z must also appear linearly. For example, the equation

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z^2 + x^2$$

is linear, whereas the equation

$$x\frac{dz}{dx} = z^2 + x^2$$

is not.
The method of solving linear equations of the form (2.2.3.1) is contained in Theorem 2.

Theorem 2: The general solution of the linear partial differential equation

$$Pp + Qq = R \qquad (2.2.3.1)$$

is

$$F(u,v) = 0 \qquad (2.2.3.3)$$

where F is an arbitrary function and $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ form a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \qquad (2.2.3.4)$$

We shall prove this theorem in two stages;
(a) We shall show that all integral surfaces of the equation (2.2.3.1) are generated by the integral curves of the equations (2.2.3.4);
(b) and then we shall prove that all surfaces generated by integral curves of the equations (2.2.3.4) are integral surfaces of the equation (2.2.3.2).

(a) If we are given that $z = f(x, y)$ is an integral surface of the partial differential equation (2.2.3.1), then the normal to this surface has direction cosines proportional to $(p, q-1)$ and the differential equation (2.2.3.2) is no more than an analytical statement of the fact that this normal is perpendicular to the direction defined by the direction ratios $(P, Q, R)$. In other words, the direction $(P, Q, R)$ is tangential to the integral surface $z = f(x, y)$

If therefore, we start from an arbitrary point M on the surface and move in such a way that the direction of motion is always $(P, Q, R)$, we trace out an integral curve of the equation (2.2.3.4) and since P, Q and R are assumed to be unique, there will be only one such curve through M. Further, since (P, Q, R) is always tangential to the surface, we never leave the surface. In other words, this integral curve of the equations (2.2.3.4) lies completely on the surface.

We have therefore shown that through each point M of the surface there is one and only one integral curve of the equations (2.2.3.4) and that this curve lies entirely on the surface. That is, the integral surface of the equation (2.2.3.1) is generated by the integral curves of the equations (2.2.3.4).

(b) Second, if we are given that the surface $z = f(x, y)$ is generated by integral curves of the equations (2.2.3.4), then we notice that its normal at a general point $(x, y, z)$ which is in the direction $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ will be perpendicular to the direction $(P, Q, R)$ of the curves generating the surface. Therefore

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} - R = 0$$

which is just another way of saying that $z = f(x, y)$ is an integral surface of equation (2.2.3.1).

To complete the proof of the theorem we have still to prove that any surface generated by the integral curves of the equations (2.2.3.4) has an equation of the form (2.2.3.3). Let any curve on the surface which is not a particular member of the system.

$$u(x, y, z) = c_1, \qquad v(x, y, z) = c_2 \qquad (2.2.3.5)$$

have equation

$$\phi(x, y, z) = 0, \qquad \varphi(x, y, z) = 0 \qquad (2.2.3.6)$$

If the curve (2.2.3.5) is a generating curve of the surface, it will intersect the curve (2.2.3.6). The condition that it should do so will be obtained by eliminating x, y and z from the four equations (2.2.3.5) and (2.2.3.6). This will be a relation of the form

$$F(c_1, c_2) = 0 \qquad (2.2.3.7)$$

between the constants $c_1$ and $c_2$. The surface is therefore generated by curves (2.2.3.5) which obey the condition (2.2.3.7) and will therefore have an equation of the form

$$F(u, v) = 0 \qquad (2.2.3.3)$$

Conversely, any surface of the form (2.2.3.3) is generated by integral curves (2.2.3.5) of the equations (2.2.3.4), for it is that surface generated by those curves of the system (2.2.3.5) which satisfy the relation (2.2.3.7).

This completes the pro of the theorem.

We have used a geometrical method of proof to establish this theorem because it seems to show most clearly the relation between the two equations (2.2.3.1) and (2.2.3.4).

## 2.2.4. Integral Surfaces Passing through a Given Curve

Earlier we studied a method of finding the general solution of a linear differential equation. We shall now indicate how such a general solution may be used to determine the integral surface which passes through a given curve. We shall suppose that we have found two solutions

$$u(x, y, z) = c_1, \qquad v(x, y, z) = c_2 \qquad (2.2.4.1)$$

of the auxiliary equations (2.2.3.4). Then, as we saw in that section, any solution of the corresponding linear equation is of the form

$$F(u, v) = 0 \qquad (2.2.4.2)$$

arising from a relation

$$F(c_1, c_2) = 0 \qquad (2.2.4.3)$$

between the constants $c_1$ and $c_2$. The problem we have to consider is that of determining the function F in special circumstances.

If we wish to find the integral surface which passes through the curve c whose parametric equations are

$$x = x(t), \qquad y = y(t), \qquad z = z(t)$$

where t is a parameter, then the particular solution (2.2.4.1) must be such that

$$u\{x(t), y(t), z(t)\} = c_1, \qquad v\{x(t), y(t), z(t)\} = c_2$$

We therefore have two equations from which we may eliminate the single variable t to obtain a relation of the type (2.2.4.3). The solution we are seeking is then given by (2.2.4.2).

## 2.2.5. Surfaces Orthogonal to a Given System of Surfaces

An interesting application of the theory of linear partial differential equations of the first order is to the determination of the systems of surfaces orthogonal to a given system of surfaces. Suppose we are given a one-parameter family of surfaces characterized by the equation

$$f(x, y, z) - c \qquad (2.2.5.1)$$

and that we wish to find a system of surfaces which cut each of these given surfaces at right angles.

The normal at the point $(x, y, z)$ to the surface of the system (2.2.5.1) which passes through that point is the direction given by the direction ratios

$$(P, Q, R) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \qquad (2.2.5.2)$$

If the surface with equation

$$z = \phi(x, y) \qquad (2.2.5.3)$$

cuts each surface of the given system orthogonally, then its normal at the point $(x, y, z)$ which is in the direction

$$\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$$

is perpendicular to the direction $(P, Q, R)$ of the normal to the surface of the set (2.2.5.1) at that point. We therefore have the linear partial differential equation

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R \qquad (2.2.5.4)$$

for the determination of the surfaces (2.2.5.3). Substituting from equation (2.2.5.2), we see that this equation is equivalent to

$$\frac{\partial f}{\partial x}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial z}{\partial y} = \frac{\partial f}{\partial z}$$

Conversely, any solution of the linear partial differential equation (2.2.5.4) is orthogonal to every surface of the system characterised by equation (2.2.5.1), for (2.2.5.4) simply states that the normal to any solution of (2.2.5.4) is perpendicular to the normal to that member of the system (2.2.5.1) which passes through the same point.

The linear equation (2.2.5.4) is therefore the general partial differential equation determining the surface orthogonal to members of the system (2.2.5.1); i.e., the surfaces orthogonal to the system (2.2.5.1) are the surfaces generated by the integral curves of the equations

$$\frac{dx}{\partial f/\partial x}=\frac{dy}{\partial f/\partial y}=\frac{dz}{\partial f/\partial z} \tag{2.2.5.5}$$

## 2.3. Non-linear first order PDE

### 2.3.1. Equations of first order

We turn now to the more difficult problem of finding the solutions of the partial differential equation

$$F(x,y,z,p,q)=0 \tag{2.3.1.1}$$

in which the function F is not necessarily linear in p and q.
The partial differential equation of the two parameter system is of the form

$$f(x,y,z,a,b)=0 \tag{2.3.1.2}$$

Any envelope of the system (2.3.1.2) touches at each of its points a member of the system. It posses therefore the same set of values $(x,y,z,p,q)$ as the particular surface, so that it must also be a solution of the differential equation. In this way we are led to three classes of integrals of a partial differential equation of the type (2.3.1.1).

(a) two parameter systems of surfaces $f(x,y,z,a,b)=0$
such an integral is called a complete integral.

(b) If we take any one-parameter sub system $f(x,y,z,a,\phi(a))=0$ of the system (2.3.1.2) and form its envelope, we obtain a solution of equation (2.3.1.1). When the function $\phi(a)$ which defines this subsystem is arbitrary, the solution obtained is called the general integral of (2.3.1.1) corresponding t the complete integral (2.3.1.2). When a definite function $\phi(a)$ is used, we obtain a particular case of the general integral.

(c) If the envelope of the two-parameter system (2.3.1.2) exists, it is also a solution of the equation (2.3.1.1); it is called the singular integral of the equation.

We can illustrate these three kinds of solution with reference to the partial differential equation

(2.3.1.3)

$$z^2(1+p^2+q^2)=1$$

We have

(2.3.1.4)

$$(x-a)^2+(y-a)^2+z^2=1$$

was a solution of this equation with arbitrary a and b. Since it contains two arbitrary constants, the solution (2.3.1.4) is thus a complete integral of the equation (2.3.1.3).

Putting b=a in equation (2.3.1.4), we obtain the one parameter subsystem

$$(x-a)^2+(y-a)^2+z^2=1$$

whose envelope is obtained by eliminating a between this equation and

x+y-2a=0

(2.3.1.5)

so that it has equation $(x-y)^2+2z^2=2$

Differentiating both sides of this equation with respect to x and y respectively, we obtain the relations

2zp=y-x, 2zq=x-y

from which it follows immediately that (2.3.1.5) is an integral surface o the equation (2.3.1.3). It is a solution of type (b); i.e. it is a general integral of the equation (2.3.1.3)

The envelope of the two-parameter system (2.3.1 3) is obtained by eliminating a and b from equation (2.3.1.4) and the two equations

x-a=0 y-b=0

i.e., the envelope consists of the pair of planes $z=\pm 1$. It is readily verified that these planes are integral surfaces of the equation (2.3.1.3); since they are of type (c) they constitute the singular integral of the equation.

It should be noted that, theoretically, it is always possible to obtain different complete integrals which are not equivalent to each other. i.e., which cannot be obtained from one another merely by a change in the choice of arbitrary constants. When, however, one complete integral has been obtained, every other solution, including every other complete integral, appears among the solutions of type (b) and (c) corresponding to the complete integral we have found.

To illustrate both these points we note that

$$(y-mx-c)^2=(1+m^2)(1-z^2) \qquad (2.3.1.6)$$

is a complete integral of equation (2.3.1.3), since it contains two arbitrary constants m and c, and it cannot be derived from the complete integral (2.3.1.4) by a simple change in the values of a and b. It can be readily shown, however, that the solution (2.3.1.6) is the envelope of the one-parameter sub system of (2.3.1.4) obtained by taking b=ma+c.

## 2.3.2. Compatible system of first order equations

We shall next consider the condition to be satisfied in order that every solution of the first-order partial differential equation

(2.3.2.1)

f (x,y,z,p,q)=0

is also a solution of the equation (2.3.2.2)

g(x,y,z,p,q)=0

When such a situation arises, the equations are said to be compatible.

If (2.3.2.3)

$$J = \frac{\partial(f,g)}{\partial(p,q)} \neq 0$$

we can solve equations (2.3.2.1) and (2.3.2.2) to obtain the explicit expressions (2.3.2.4)

$$p = \phi(x,y,z), \qquad q = \psi(x,y,z)$$

for p and q. The condition that the pair of equations (2.3.2.1) and (2.3.2.2) should be compatible reduces then to the condition that the system of equations (2.3.2.4) should be complexly integrable, i.e., that the equation

$$\phi dx + \psi dy - dz = 0$$

should be integrable. We have that the condition that this equation is integrable is

$$\phi(-\psi_z) + \psi(\phi_z) - (\psi_x - \phi_y) = 0$$

which is equivalent to

$$\psi_x + \phi\psi_z = \phi_y + \psi\phi_z \qquad (2.3.2.5)$$

Substituting from equations (2.3.2.4) into equation (2.3.2.1) and differentiating with regard to x and z, respectively, we obtain the equations

$$f_x + f_p\phi x + f_p\psi_x = 0$$

$$f_z + f_p\phi x + f_p\psi_z = 0$$

From which it is readily deduced that

$$f_x + \phi f_z + f_p(\phi_x + \phi\phi_z) + f_p(\psi_x + \phi\psi_z) = 0$$

Similarly we may deduce from equation (2.3.2.2) that

$$g_x + \phi g_z + g_p(\phi_x + \phi\phi_z) + g_p(\psi_x + \phi\psi_z) = 0$$

Solving these equations, we find that

$$\psi x + \phi\psi z = \frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)} + \phi\frac{\partial(f,g)}{\partial(z,p)}\right\} \qquad (2.3.2.6)$$

where J is defined as equation (2.3.2.3).

If we had differentiated the given pair of equations with respect to y and z, we should have obtained

$$\phi_y + \psi\phi_z = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)} + \psi\frac{\partial(f,g)}{\partial(z,p)}\right\} \qquad (2.3.2.7)$$

so that, substituting from equations (2.3.2.6) and (2.3.2.7) into equation (2.3.2.5) and replacing $\phi, \psi$ by p, q respectively, we see that the condition that the two conditions should be compatible is that

$$[f,g] = 0 \qquad (2.3.2.8)$$

where $[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(z,q)}$

(2.3.2.9)

Example. Show that the equation

$$xp = yq, \qquad z(xp + yq) = 2xy$$

Are compatible and solve them.

Solution: In this example we may take

$f = xp = yq, g = z(xp + yq) - 2xy$ so that

$$\frac{\partial(f,g)}{\partial(x,p)} = 2xy, \quad \frac{\partial(f,g)}{\partial(z,p)} = -x^2 p, \quad \frac{\partial(f,g)}{\partial(y,q)} = -2xy$$

$$\frac{\partial(f,g)}{\partial(z,q)} = xyp$$

From which it follows that

$$[f,g] = xp(yq - xp) = 0$$

since xp=yq. The equations are therefore compatible.

It is readily shown that $p \backslash y/z, q = x/z$ so that we have to solve

zdz=ydx+xdy

which has solution

$z^2 = c_1 + 2xy$

where $c_1$ is a constant.

## 2.3.2.1. Self Assessment Questions:

1. Show that the equations f(x,y,p,q)=0, g(x,y,p,q)=0 are compatible if

$$\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)}=0$$

Verify that the equations p=P(x,y), q=Q(x,y) are compatible if

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

2. If $u_1=\partial u/\partial x, u_2=\partial u/\partial y, u_3=\partial u/\partial z$, show that the equations

$f(x,y,z,u_1,u_2,u_3)=0, \quad g(x,y,z,u_1,u_2,u_3)=0$

are compatible if

$$\frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(y,u_2)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0$$

### 2.3.3. Charpit's Method

A method of solving the partial differential equation

$$f(x,y,z,p,q)=0 \tag{2.3.3.1}$$

due to Charpit, is based on the considerations of the last section. The fundamental idea in Charpit's method is the introduction of a second partial differential equation of the first order

$$g(x,y,z,p,q)=0 \tag{2.3.3.2}$$

which contains an arbitrary constant a and which is such that

(a) Equations (2.3.3.1) and (2.3.3.2) can be solved to give

$p=p(x,y,z,a), \qquad q=q(x,y,z,a),$

(b) The equation

$$dz=p(x,y,z,a)dx+q(x,y,z,a)dy \tag{2.3.3.3}$$

is integrable.

When such a function g has been found, the solution of equation (2.3.3.3)

$$F(x,y,z,a,b)=0 \tag{2.3.3.4}$$

containing two arbitrary constants a, b will be a solution of equation (2.3.3.1).

It will be seen that the equation (2.3.3.4) is a complete integral of equation (2.3.3.1).

The main problem then is the determination of the second equation (2.3.3.2), but this has already been solved in the last section, since we need only seek an equation g=0 compatible with the given

$$\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)}=0$$

Verify that the equations p=P(x,y), q=Q(x,y) are compatible if

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$$

2. If $u_1=\partial u/\partial x, u_2=\partial u/\partial y, u_3=\partial u/\partial z$, show that the equations

$f(x,y,z,u_1,u_2,u_3)=0, \quad g(x,y,z,u_1,u_2,u_3)=0$

are compatible if

$$\frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(y,u_2)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0$$

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$$dz=p(x,y,z,a)dx+q(x,y,z,a)dy \tag{2.3.3.3}$$

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The main problem then is the determination of the second equation (2.3.3.2), but this has already been solved in the last section, since we need only seek an equation g=0 compatible with the given

## 2.3.3.1. Self Assessment Questions:

Find the complete integrals of the equations

1. $(p^2 + q^2)y = qz$
2. $p = (z + qy)^2$

## 2.3.4. Special types first order equations

In this section we shall consider some special types of first-order partial differential equations whose solution may be obtained easily by Charpit's method.

(a) Equations Involving only p and q. For equations of the type

$$f(p,q)=0 \tag{2.3.4.1}$$

Charpit's equations reduce to

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

An obvious solution of these equations is

$$p=a \tag{2.3.4.2}$$

the corresponding value of q being obtained from (2.3.4.1) in the form

$$f(a,q)=0 \tag{2.3.4.3}$$

so that $q=Q(a)$

a constant. The solution of the equation is then

$$z=ax+Q(a)y+b \tag{2.3.4.4}$$

where b is a constant.

We have chosen the equation $dp=0$ to provide our second equation. In some problems the amount of computation involved is considerably reduced if we take instead $dq=0$, leading to $q=a$.

(b) Equations Not involving the Independent Variables. If the partial differential equation is of the type

$$f(z,p,q)=0 \tag{2.3.4.5}$$

Charpit's equations take the forms

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

The last of which leads to the relation

$$p=aq \tag{2.3.4.6}$$

Solving (2.3.4.5) and (2.3.4.6), we obtain expressions for p, q from which a complete integral follows immediately.

(c) Separable Equations. We say that a first-order partial differential is separable if it can be written in the form f(x,p)=g(y,q) (2.3.4.7)

For such an equation Charpit's equations become

$$\frac{dx}{f_p}=\frac{dy}{-g_q}=\frac{dz}{pf_p-qg_q}=\frac{dp}{-f_x}=\frac{dq}{-g_y}$$

So that we have an ordinary differential equation

$$\frac{dp}{dx}+\frac{f_x}{f_p}=0$$

- In x and p which may be solved to give p as a function of x and an arbitrary constant a. Writing this equation in the form $f_p dp+f_x dx=0$ we see that its solution is f(x,p)=a.. Hence we determine p, q from the relations

(2.3.4.8)

(x,p)=a, g(y,q)=a

and then proceed as in the general theory.

(d) **Clariaut Equations.** A first order partial differential equation is said to be of Clairaut type if it can be written in the form

z=px+qy+f(p,q) (2.3.4.9)

The corresponding Charpit equations are

$$\frac{dx}{x+f_p}=\frac{dy}{y+f_q}=\frac{dz}{px+qy+pf_p+qf_q}=\frac{dp}{0}=\frac{dq}{0}$$

So that we may take p=a, q=b. If we substitute these values in (2.3.4.9), we get the complete integral

z=ax+by+f(a,b) (2.3.4.10)

as is readily verified by direct differentiation.

## 2.3.4.1.Self Assessment Questions:

Find complete integrals of the equations

1. $p+q=pq$
2. $z=p^2-q^2$
3. $zpq=p+q$
4. $p^2q(x^2+y^2)=p^2+q$
5. $p^2q^2+x^2y^2=x^2q^2(x^2+y^2)$

## 2.3.5. Jacobi's Method

Another method, due to Jacobi, of solving the partial differential equation

F(x,y,z,p,q)=0 (2.3.5.1)

depends on the fact that if

u(x,y,z)=0 (2.3.5.2)

is a relation between x, y and z, then

$$p = -\frac{u_1}{u_3}, \qquad q = -\frac{u_2}{u_3} \qquad (2.3.5.3)$$

where $u_i$ denotes $\partial u / \partial x_i\,(i = 1,2,3)$. If we substitute from equations (2.3.5.3) into equation (2.3.5.1), we obtain a partial differential equation of the type

$$f\,(x,y,z,u_1,u_2,u_3)=0 \qquad (2.3.5.4)$$

in which the new dependent variable u does not appear.

The fundamental idea of Jacobi's method is the introduction of two further partial differential equations of the first order.

$$g\,(x,y,z,u_1,u_2,u_3)=0, \quad h\,(x,y,z,u_1,u_2,u_3)=0 \qquad (2.3.5.5)$$

involving two arbitrary constants a and b and such that

(a) Equations (2.3.5.4) and (2.3.5.5) can be solved for

(b) The equation

$$du=u_1dx + u_2dy + u_3du \qquad (2.3.5.6)$$

obtained from these values of $u_1$, $u_2$, $u_3$ is integrable.

When these functions have been found, the solution of equation (2.3.5.6) containing three arbitrary constants will be a complete integral of (2.3.5.4). The three constants are necessary if the given equation is (2.3.5.4); when, however, the equation is given in the form (2.3.5.1), we need only two arbitrary constants in the final solution. By taking different choices of our third arbitrary constant we get different complete integrals of the given equation.

As in Charpit's method, the main difficulty is in the determination of the auxiliary equations (2.3.5.5). We have, in effect, to find two equations which are compatible with (2.3.5.4). Now g and h would therefore have to be solutions of the linear partial differential equation

$$fu_1\frac{\partial g}{\partial x} + fu_2\frac{\partial g}{\partial y} + fu_3\frac{\partial g}{\partial z} - fx\frac{\partial g}{\partial u_1} - fy\frac{\partial g}{\partial u_2} - fz\frac{\partial g}{\partial u_3} = 0 \qquad (2.3.5.7)$$

which has subsidiary equations

$$\frac{dx}{fu_1} = \frac{dy}{fu_2} = \frac{dz}{fu_3} = \frac{du_1}{-fx} = \frac{du_2}{-fy} = \frac{du3}{-fz} \qquad (2.3.5.8)$$

The procedure is then the same as in Charpit's method.

The advantage of the Jacobi method is that it can readily be generalized. If we have to solve an equation of the type

$$f_1(x_1, x_2, \ldots x_n, u_1, \ldots, u_n) = 0 \qquad (2.3.5.9)$$

where $u_i$ denotes $\partial u \partial x_i\,(i = 1,2\ldots,n)$, then we find n-1 auxiliary functions $f_2, f_3, \ldots, f_n$ from the subsidiary equations

$$\frac{dx_1}{fu_1} = \frac{dx_2}{fu_2} = \ldots = \frac{dx_n}{fu_n} = \frac{du_1}{-fx_1} = \frac{du_2}{-fx_2} = \ldots = \frac{du_n}{-fx_n}$$

involving n-1 arbitrary constants. Solving these for $u_1, u_2, \ldots, u_n$ we determine u by integrating the Pfaffian equation

$$du = \sum_{i=1}^{n} u_i dx_i$$

the solution so obtained containing n arbitrary constants. On the other hand, Charpit's method cannot be generalized directly.

### 2.3.5.1.Self Assessment Questions:

Solve the problems by Jacobi's method

1. $(p^2+q^2)y=qz$
2. $p=(z+qy)^2$
3. $z^2=pqxy$
4. $xp+3yq=2(z-x^2q^2)$
5. $px^5-4q^3x^2+6x^2z-2=0$

## 2.4. Second order PDE

### 2.4.1. Linear equations with constant co-efficient

We shall now consider the solution of a very special type of linear partial differential equation, that with constant coefficients. Such an equation can be written in form

$$F(D, D')z=f(x,y) \qquad (2.4.1.1)$$

where F(D,D') denotes a differential operator of the type

$$F(D,D') = \sum_r \sum_s c_{rs} D^r D'^s \qquad (2.4.1.2)$$

in which the quantities $c_{rs}$, are constants, and $D = \partial / \partial x, D' = \partial / \partial y$. The most general solution i.e., one containing the correct number of arbitrary elements, of the corresponding homogeneous linear partial differential equation

$$F(D,D')z = 0 \qquad (2.4.1.3)$$

is called the complementary function of the equation (2.4.1.1), just as in the theory of ordinary differential equations. Similarly any solution of the equation (2.4.1.1) is called a particular integral of (2.4.1.1).

As in the theory of linear ordinary differential equations, the basic theorem is:

**Theorem 1.** If u is the complementary function and $z_1$ a particular integral of a linear partial differential equation, then $u+z_1$ is a general solution of the equation.

The proof of this theorem is obvious. Since the equations (2.4.1.1) and (2.4.1.3) are of the same kind, the solution $u+z_1$ will contain the correct number of arbitrary elements to qualify as a general solution of (2.4.1.1). Also

$$F(D,D')u = 0, \qquad F(D,D')z_1 = f(x,y)$$

so that $\quad F(D,D')(u+z_1) = f(x,y)$

showing that $u+z_1$ is in fact a solution of equation (2.4.1.1). This completes the proof.

Another result which is used extensively in the solution of differential equations is:

**Theorem 2.** If $u_1, u_2, \ldots, u_n$ are solutions of the homogeneous linear partial differential equation F(D, D')z=0, then

$$\sum_{r=1}^{n} c_r u_r$$

where the $c_r$'s are arbitrary constants, is also a soiution.

The proof of this theorem is obvious. Since

$$F(D,D')(c_r u_r) = c_r F(D,D')u_r$$

and $\quad F(D,D')\sum_{r=1}^{n} v_r = \sum_{r=1}^{n} F(D,D')v_r$

for any set of functions $v_r$. Therefore

$$F(D,D')\sum_{r=1}^{n} c_r u_r = \sum_{r=1}^{n} F(D,D')(c_r u_r)$$

$$= \sum_{r=1}^{n} c_r F(D,D')u_r$$

$$= 0$$

We classify linear differential operators F(D,D') into two main types, which we shall treat separately. We say that:

(a) F(D,D') is reducible if it can be written as the product of linear factors of the form D+aD'+b with a, b constants;

(b) F(D,D') is irreducible if it cannot be written.

For example, the operator

$D^2-D'^2$

Which can be written in the form

(D+D')(D-D')

is reducible, whereas the operator

$D^2-D'$

which cannot be decomposed into linear factors, is irreducible.

(a) Reducible Equations: The starting point of the theory of reducible equations is the result

**Theorem 3.** If the operator F(D,D') is reducible, the order in which the linear factor occur is unimportant.
The theorem will be proved if we can show that

$$(\alpha_r D+\beta_r D'+\gamma_r)(\alpha_s D+\beta_s D'+\gamma_s)=(\alpha_s D+\beta_s D'+\gamma_s)(\alpha_r D+\beta_r D'+\gamma_r) \quad (2.4.1.4)$$

for any reducible operator can be written in the form

$$F(D,D')=\prod_{r=1}^{n}(\alpha_r D+\beta_r D'+\gamma_r) \quad (2.4.1.5)$$

and the theorem follows at once. The proof of (2.4.1.4) is immediate, since both sides are equal to

$$\alpha_r\alpha_s D^2+(\alpha_s\beta_r+\alpha_r\beta_s)DD'+\beta_r\beta_s$$
$$D'^2+(\gamma_s\alpha_r+\gamma_r\alpha_s)D+(\gamma_s\beta_r+\gamma_r\beta_s)D'+\gamma_r\gamma_s.$$

**Theorem 4.** If $\alpha_r D+\beta_r D'+\chi_r$ is a factor of F(D,D') and $\phi_r(\xi)$ is an arbitrary function of the single variable $\xi$, then if $\alpha_r \neq 0$,

$$u_r=\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x-\alpha_r y)$$

is a solution of the equation $F(D,D')z=0$

By direct differentiation we have

$$Du_r=-\frac{\gamma_r}{\alpha_r}u_r+\beta_r\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi'(\beta_r x-\alpha_r y)$$

$$D'u_r=-\alpha_r\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi'(\beta_r x-\alpha_r y)$$

So that $(\alpha_r D+\beta_r D'+\gamma_r)u_r=0$ (2.4.1.6)
Now by Theorem 3

$$F(D,D')u_r=\left\{\prod_{s=1}^{n}{}'(\alpha_s D+\beta_s D'+\gamma_s)\right\}(\alpha_r D+\beta_r D'+\gamma_r)u_r \quad (2.4.1.7)$$

the prime after the product denoting that the factor corresponding to s=r is omitted. Combining equations (2.4.1.6) and (2.4.1.7) we see that

$$F(D,D')u_r=0$$

which proves the theorem.

**Theorem 5.** If $\beta_r D'+\gamma_r$ is a factor of F(D, D') and $\phi_r(\xi)$ is an arbitrary function of the single variable $\xi$, then if $\beta_r \neq 0$

$$u_r=\exp\left(-\frac{\gamma_r y}{\beta_r}\right)\phi_r(\beta_r x)$$

is a solution of the equation $F(D,D')z=0$

In the decomposition of $F(D,D')$ into linear factors we may get multiple factors of the type $(\alpha_r D+\beta_r D'+\gamma_r)^n$. The solution corresponding to a factor of this type can obtained by a simple application of Theorem 4 and 5. For example if n=2 we wish to find solutions of the equation

$$(\alpha_r D+\beta_r D'+\gamma_r)^2 z=0 \qquad (2.4.1.8)$$

If we let

$$Z=(\alpha_r D+\beta_r D'+\gamma_r)z$$

Then $\quad (\alpha_r D+\beta_r D'+\gamma_r)Z=0$

Which according to Theorem 4 has solution

$$Z=\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\phi_r(\beta_r x-\alpha_r y)$$

If $\alpha_r \neq 0$. To find the corresponding function z we have therefore to solve the first-order linear partial differential equation

$$\alpha_r\frac{\partial z}{\partial x}+\beta_r\frac{\partial z}{\partial y}+\gamma_r z=e^{-\gamma_r x/\alpha_r}\phi_r(\beta_r x-\alpha_r y) \qquad (2.4.1.9)$$

The auxiliary equations are

$$\frac{dx}{\alpha_r}=\frac{dy}{\beta_r}=\frac{dz}{-\gamma_r z+e^{-\gamma_r x/\alpha_r}\phi_r(\beta_r x-\alpha_r y)}$$

With solution

$$\beta_r x-\alpha_r y=c_1$$

Substituting this in the auxiliary equations, we get the

$$\frac{dx}{\alpha_r}=\frac{dz}{-\gamma_r z+e^{-\gamma_r x/\alpha_r}\phi_r(c_1)}$$

Which is a first order linear equation with solution

$$z=\frac{1}{\alpha_r}\{\phi_r(c_1)x+c_2\}e^{-\gamma_r x/\alpha_r}$$

Equation (2.4.1.9) and hence equation (2.4.1.8) therefore has solution

$$z=\{x\phi_r(\beta_r x-\alpha_r y)+\psi_r(\beta_r x-\alpha_r y)\}e^{-\gamma_r x/\alpha_r}$$

Where the functions $\phi_r, \psi_r$ are arbitrary.

**Theorem 6.** If $(\alpha_r D+\beta_r D'+\gamma_r)^n(\alpha_r \neq 0)$ is a factor of F(D, D') and if the functions $\phi_{r1},\ldots\ldots,\phi_{rn}$ are arbitrary, then

$$\exp\left(-\frac{\gamma_r y}{\beta_r}\right)\sum_{s=1}^{n}x^{s-1}\phi_{rs}(\beta_r x-\alpha_r y)$$

is a solution of F(D, D')z=0.

Similarly the generalisation of Theorem 5 is:

**Theorem 7.** If $(\beta_r D' + \gamma_r)^m$ is a factor of F(D, D') and if the functions $\phi_{r1}, \ldots\ldots, \phi m$ are arbitrary, then

$$\exp\left(-\frac{\gamma_r y}{\beta_r}\right)\sum_{s=1}^{m} x^{s-1}\phi_{rs}(\beta_r x)$$

Is a solution of F(D, D')=0

We are now in a position to state the complementary function of the equation (2.4.1.1) when the operator is reducible. As a result of Theorem 4 and 6, we see that if

$$F(D,D') = \prod_{r=1}^{n}(\alpha_r D + \beta_r D' + \gamma_r)^{mr} \qquad (2.4.1.10)$$

and if none of the is zero, then the corresponding complementary function is

$$u = \sum_{r=1}^{n}\exp\left(-\frac{\gamma_r x}{\alpha_r}\right)\sum_{s=1}^{m} x^{s-1}\phi_{rs}(\beta_r x - \alpha_r y) \qquad (2.4.1.11)$$

where the functions $\phi_{rs}(s = 1,\ldots, n_r; r = 1,\ldots\ldots, n)$ are arbitrary. If some of the $\alpha's$ are zero the necessary modifications to the expressions (2.4.1.11) can be made by means of Theorem 5. From equation (2.4.1.10) we see that the order of equation (2.4.1.3) is $m_1+m_2+\ldots+m_n$; since the solution (2.4.1.11) contains the same number of arbitrary functions, it has the correct number and is thus the complete complementary function.

(b) Irreducible equations: When the operator F(D,D') is irreducible, it is not always possible to find a solution with the full number of arbitrary functions, but it is possible to constructs solutions which contain as many arbitrary constants as we wish. The method of deriving such solution depends on a theorem which we shall now prove. This theorem is true for reducible as well as irreducible operators, but it is only in the irreducible case that we make use of it.

**Theorem 7.** $F(D,D')e^{ax+by} = F(a,b)e^{ax+by}$

The proof of this theorem follows from the fact that F(D, D') is made up of terms of the type

$$c_{rs} D^r D'^s$$

and $D^r(e^{ax+by}) = a^r e^{ax+by}$, $\qquad D'^s(e^{ax+by}) = b^s e^{ax+by}$

so that $(c_{rs} D^r D'^s)(e^{ax+by}) = (c_{rs} a^r b^s e^{ax+by}$

The theorem follows by recombining the terms of the operator

A similar result which is used in determining particular integrals is:

**Theorem 8.** $F(D,D')\{e^{ax+by}\phi(x,y)\} = e^{ax+by}F(D+a,D'+b)\phi(x,y)$
The proof is direct, making use of Leibnitz's theorem for the r th derivative of a product to show that

$$D^r(e^{ax}\phi) = \sum_{\rho=0}^{r} {}^rC_\rho (D^\rho e^{ax})(D^{r-\rho}\phi)$$

$$= e^{ax}\left(\sum_{\rho=0}^{r} {}^rC_\rho a^\rho D^{r-\rho}\right)\phi$$

$$= e^{ax}(D+a)^r\phi$$

To determine the complementary function of an equation of the type (2.4.1.1) we split the operator F(D, D') into factors. The reducible factors are treated by method (a). The irreducible factors are treated as follows. From theorem 8 we see that $e^{ax+by}$ is a solution of the equation

$$F(D, D')z=0 \qquad (2.4.1.14)$$

Provided F(a,b)=0, so that

$$z = \sum_{\rho=0} c_r \exp(a_r x + b_r y) \qquad (2.4.1.15)$$

in which $a_r$, $b_r$, $c_r$ are all constants, is also a solution provided that are connected by the relation

$$F(a_r,b_r)=0 \qquad (2.4.1.16)$$

In this way we can construct a solution of the homogeneous equation (2.4.1.14) containing as many arbitrary constants as we need. The series (2.4.1.15) need not be finite, but if it is infinite, it is necessary that it should be uniformly convergent if it has to be, in fact, a solution of equation (2.4.1.14). The discussion of the convergence of such a series is difficult, involving as it does the coefficients $c_r$, the pairs $(a_r, b_r)$ and the values of the variables x and y.

## 2.4.1.1.Self Assessment Questions:

## Solve the equations

1. $r+s-2t=e^{x+y}$
2. $r-s+2q-z=x^2y^2$
3. $r+s-2t-p-2q=0$

## 2.4.2. Characteristic curves

We shall now consider briefly the Cauchy problem for the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad (2.4.2.1)$$

in which R,S, and T are functions of x and y only. In other words, we wish to consider the problem of determining the solution of equation (2.4.2 .1) such that on a given space curve $\Gamma$ it takes on prescribed values of z and $\partial z / \partial n$ ,where n is distance measured along the normal to the curve. This latter set of boundary conditions is equivalent to assuming that the values of x,y,z,p,q are determined on the curve, but it should be noted that the values of the partial derivatives p and q cannot be assigned arbitrarily along the curve. For if we take the freedom equations of the curve $\Gamma$ to be

... (2.4.2.2)

$$x = x_0(\tau), \qquad y = y_0(\tau), \qquad z = z_0(\tau)$$

then we must have all points of $\Gamma$ the relation

... (2.4.2.3)

$$\dot{z}_0 = p_0 \dot{x}_0 + q_0 \dot{y}_0$$

(where $\dot{z}_0$ denotes $dz_0 / dt$ ,etc.) showing that $p_0$ and $q_0$ are not independent. The Cauchy problem is therefore that of finding the solution of equation (1) passing through the integral strip of the first order formed by the planar elements $(x_0, y_0, z_0, p_0, q_0)$ of the curve $\Gamma$ .

At every point of the integral strip $p = p_0(\tau)$ , $q_0 = q_0(\tau)$, so that if we differentiate these equations with respect to $\tau$ , we obtain the relations

$$\dot{p}_0 = r\,\dot{x}_0 + s\,\dot{y}_0, \qquad \dot{q}_0 = s\,\dot{x}_0 + t\,\dot{y}_0$$

If we solve the three equations (1) and (4) for r, s, t, we find that

$$\frac{r}{\Delta_1} = \frac{-s}{\Delta_2} = \frac{t}{\Delta_3} = \frac{-1}{\Delta}$$

where

$$\Delta_1 = \begin{vmatrix} S & T & f \\ \dot{y}_0 & 0 & -\dot{p}_0 \\ \dot{x}_0 & \dot{y}_0 & -\dot{q}_0 \end{vmatrix}, \text{etc} \qquad \text{and} \qquad \Delta = \begin{vmatrix} R & S & T \\ \dot{x}_0 & \dot{y}_0 & 0 \\ 0 & \dot{x}_0 & \dot{y}_0 \end{vmatrix}$$

If $\Delta \neq 0$ ,we can therefore easily calculate the expressions for the second –order derivatives $r_0, s_0$ and $t_0$ ,along the curve $\Gamma$ .

The third order partial differential coefficients of z can similarly be calculated at every point of $\Gamma$ by differentiating equation (2.4.2.1) with respect to x and y , respectively, making use of the relations

$$r_0 = z_{xxx}\dot{x}_0 + z_{xxy}\dot{y}_0$$

etc., and solving as in the previous case.

Proceeding in this way, we can calculate the partial derivatives of every order at the points of the curve $\Gamma$ . The value of the function z at neighbouring points can therefore be obtained by means of Taylor's theorem for functions of two independent variables. The Cauchy problem therefore possesses a solution as long as the determinant $\Delta$ does not vanish. In the elliptic case $4RT - S^2 > 0$, so that $\Delta \neq 0$ always holds, and the derivatives, of all orders, of z are uniquely determined. It is reasonable to conjecture that the solution so obtained is analytic in the domain of analyticity of the coefficients of the differential equation being discussed; constructing a proof of this conjecture was one of the famous problems propounded by Hilbert. The proof for the linear case was given first by Bernstein; that for the general case (2.4.2.1) was given later by Hopf and Lewy.

We must now consider the case in which the determinant $\Delta$ vanishes. Expanding $\Delta$ ,we see that this condition is equivalent to the relation

$$R\dot{y}_0^2 - S\dot{x}_0\dot{y}_0 + T\dot{x}_0^2 = 0 \qquad \ldots (2.4.2.5)$$

If the projection of the curve $\Gamma$ onto the plane z=0 is a curve $\gamma$ with equation

$$\xi(x, y) = c_0 \qquad \ldots (2.4.2.6)$$

then we find that ,as a result of differentiating with regard to $\tau$ ,

$$\xi_x \dot{x}_0 + \xi_y \dot{y}_0 = 0 \qquad \ldots (2.4.2.7)$$

Eliminating the ratio $\dot{x}_0/\dot{y}_0$ between equations (2.4.2.5) and (2.4.2.7),we find that the condition $\Delta = 0$ is equivalent to the relation

$$A(\xi_x, \xi_y) = 0 \qquad \ldots (2.4.2.8)$$

where the function $A(u,v) = Ru^2 + Suv + Tv^2$

A curve $\gamma$ in the xy plane satisfying the relation (2.4.2.8) is called a characteristic base curve of the partial differential equation (1), and the curve $\Gamma$ of which it is the projection is called a characteristic curve of the same equation. The term characteristic is applied indiscriminately to both kinds of curves, since there is usually little danger of confusion arising as a result.

From the argument of Sec.2.4.3 it follows at once that there are two families of characteristics if the given partial differential equation is hyperbolic , one family if it is parabolic, and none if it is elliptic.

As we have defined it, a characteristic is a curve such that, given values of the dependent variable and its first-order partial derivatives at all points on it ,Cauchy's problem does not posses a unique solution We shall now show that this property is equivalent to one which is of ore interest in physical applications namely, that if there is a second-order discontinuity at one point of the characteristic, it must persist at all points.

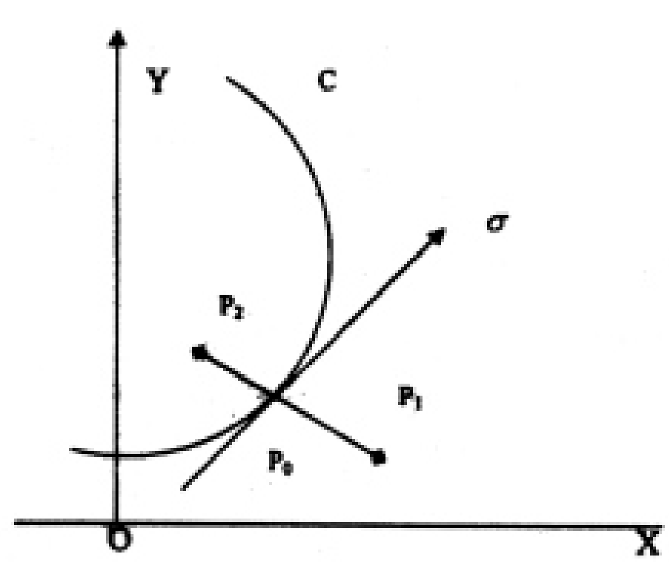
To establish this property we consider a function $\phi$ of the independent variables x and y which is continuous everywhere except at the points of the curve C whose equation is

$$\xi(x,y)=c$$

where $\xi(x,y)$ is any function (not necessarily the function $\xi$ defined above) with as many derivatives as necessary. If $P_0$ is any point on this curve and $P_1$ and $P_2$ are neighbouring points on opposite sides of the curve (Fig 1), then we define the discontinuity of the function $\phi$ at the point $P_0$ by the equation

$$[\phi]_{P_0}=\lim_{P_1,P_2\to P_0}\{\phi(P_1)-\phi(P_2)\} \qquad \dots (2.4.2.10)$$

If the element of length along the directed tangent to the curve C at the point $P_0$ is $d\sigma$, then




The tangential derivative of the function $\phi$ is defined to be

$$\frac{d\phi}{d\sigma}=\frac{\partial\phi}{\partial\sigma}\cos(\sigma,x)+\frac{\partial\phi}{\partial y}\cos(\sigma,y)$$

and it is readily shown that this is equivalent to the expression

$$\frac{d\phi}{d\sigma}=\frac{\phi_x\xi_y(P_0)-\phi_y\xi_x(P_0)}{\left\{\xi_x^2(P_0)+\xi_y^2(P_0)\right\}^{\frac{1}{2}}} \qquad \ldots (2.4.2.11)$$

The tangential derivative at $P_0$ is therefore continuous if the expression on the right hand side of this equation is continuous at $P_0$ , and we say that $d\phi/d\sigma$ is continuous on the curve C if this holds for all points $P_0$ on C.

Now let us suppose that the function z(x, y) is a solution of the equation (2.4.2.1) , where ,for simplicity ,we shall suppose that the function $f$ is linear in p and q. We shall assume in addition that the function z(x, y) is continuous and has continuous derivatives of all orders required except that its second derivatives are not all continuous at all points of the curve C defined by equation (2.4.2.9). In particular it is assumed that the first-order partial derivatives $z_x$ and $z_y$ have continuous tangential derivatives at all points of the curve C. It follows immediately from equation(2.4.2.11) that if the tangential derivative $dz_x/d\sigma$ is continuous at the point $P_0$, so also is the expression

$$z_{xx}\xi_y(P_0)-z_{xy}\xi_x(P_0)$$

Now another way of saying that a function is continuous is to say that its discontinuity is zero at the point in question. We may therefore write

$$\left[z_{xx}\right]\xi_y(P_0)-\left[z_{xy}\right]\xi_x(P_0)=0$$

By considering the other tangential derivative $dz_y/d\sigma$, we may similarly prove the relation

$$\left[z_{xy}\right]\xi_y(P_0)-\left[z_{yy}\right]\xi_x(P_0)=0$$

and hence that

$$\frac{[z_{xx}]}{\xi_x^2(P_0)} = \frac{[z_{xy}]}{\xi_x(P_0)\xi_y(P_0)} = \frac{[z_{yy}]}{\xi_y^2(P_0)} \qquad \ldots (2.4.2.12)$$

Letting each of the ratios in the equations (12) be equal to $\lambda$, we may write these equations in the form

$$[z_{xx}] = \lambda\xi_x^2(P_0), \quad [z_{xy}] = \lambda\xi_x(P_0)\xi_y(P_0),$$

$$[z_{yy}] = \lambda\xi_y^2(P_0) \qquad \ldots (2.4.2.13)$$

If we now transform the independent variables in our problem from x and y to $\xi$ and $\eta$, where $\xi$ is the function introduced through the curve C and $\eta$ is such that ,for any function $\psi(\xi,\eta), d\psi/d\sigma = \partial\psi/\partial\eta$ .The quantity $\lambda$ occurring in equation. (13) will then be a function of $\eta$ alone; we shall now proceed to determine that function.

Since $\quad z_{xx} = z_{\xi\xi}\xi_x^2 + 2z_{\xi\eta}\xi_x\eta_x + z_{\eta\eta}\eta_x^2 + z_\xi\xi_{xx} + z_\eta\eta_{xx}$

and since $z_\xi$ and $z_\eta$ are continuous ( a result of the continuity of $z_x$ and $z_y$) and $z_{\xi\eta}$ and $z_{\eta\eta}$ are tangential derivatives, we find that $[z_{xx}]$, which by definition is equal to

$$\lim_{P_1,P_2 \to P_0} \{z_{xx}(P_2) - z_{xx}(P_1)\}$$

reduces to $\quad \lim_{P_1,P_2 \to P_0} \{z_{\xi\xi}(P_2)\xi_x^2(P_2) - z_{\xi\xi}(P_1)\xi_x^2(P_1)\}$

so that $\quad [z_{xx}] = [z_{\xi\xi}]\xi_x^2(P_0)$

$$\ldots (2.4.2.14)$$

A comparison of equation (2.4.2.14) with the equation (2.4.2.13) shows that the value of the quantity $\lambda$ occurring in these equations is $[z_{\xi\xi}]$. We began by assuming that there was a discontinuity in at least one of the second derivatives; so $\lambda$ cannot be zero , and hence neither can $[z_{\xi\xi}]$ at the point $P_0$.

If we transform the equation to the new variables $\xi$ and $\eta$, we get the equation (2.4.3.4) of Sec 2.4.3,and applying the above argument to it, we see that

$$\left[z_{\xi\xi}\right]A(\xi_x,\xi_y)=0$$

showing that

$$A(\xi_x,\xi_y)=0 \qquad \dots (2.4.2.15)$$

and thereby proving that the curve C is a characteristic of the equation . If we differentiate the transformed equation with regard to $\xi$, take equation (15) into account ,and note that only the terms in $z_{\xi\xi}$ and $z_{\xi\xi\eta}$ can be discontinuous ,we can use a similar argument to show that

$$2B(\xi_x,\xi_y;\eta_x,\eta_y)\left[z_{\xi\xi\eta}\right]+\{A_\xi(\xi_x,\xi_y)-F_\zeta\}\left[z_{\xi\xi}\right]=0$$

remembering that $\left[z_{\xi\xi}\right]$ is $\lambda$ and that $\lambda$ is a function of $\eta$ alone, we see that this last equation is equivalent to the ordinary differential equation

$$\frac{d\lambda}{d\eta}=\lambda\, g(\eta)$$

which has a solution of the form

$$\lambda(\eta)=\lambda(\eta_0)\exp\{\int_{\eta_0}^{\eta} g(\zeta)d\zeta\}$$

So far we have considered only single characteristic curves; now let us consider briefly all the characteristic curves on an integral surface $\Sigma$ of the differential equation (2.4.2.1) .If the equation is hyperbolic at all points of the surface ,there are two one-parameter families of characteristic curves on $\Sigma$. It follows that two integral surfaces can touch only along a characteristic, for if the line of contact were not a characteristic, it would define unique values of all partial derivatives along its length and would therefore yield one surface, not the postulated two. Along a characteristic curve, on the other hand, this contradiction does not occur. In the case of elliptic equations, for which there are no real characteristics, the corresponding result would be that two integral surfaces cannot touch along any line.

## 2.4.2.1.Self Assessment Questions:

1. Show that the characteristics of the equation

$$Rr+Ss+Tt=f(x,y,z,p,q)$$

are invariant with respect to any transformations of the independent variables.

2. Show that characteristics of the second-order equation

$$A\frac{\partial^2 z}{\partial x^2}+2B\frac{\partial^2 z}{\partial x\partial y}+C\frac{\partial^2 z}{\partial y^2}=F(x,y,z,p,q)$$

are the same as the projections on the xy plane of the Cauchy characteristics of the first order-equation

$$Ap^2+2Bpq+Cq^2=0$$

3. In the one dimensional unsteady flow of a compressible fluid the velocity u and the density $\rho$ satisfy the equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=0, \qquad \frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}=0$$

If the law connecting the pressure $p$ with the density $\rho$ is $p=k\rho^2$, show that

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+2c\frac{\partial c}{\partial x}=0, \qquad 2\frac{\partial c}{\partial t}+2u\frac{\partial c}{\partial x}+c\frac{\partial u}{\partial x}=0$$

where $c^2=dp/d\rho$. Prove that the characteristics are given by the differential equations $dx=(u+c)dt$ and that on the characteristics $u+2c$ are constant.

If there is a family of straight characteristics $x=mt$ satisfying the differential equation $dx/dt=u+c$, prove that

$$u=\frac{2x}{3t}+\mu, \qquad c=\frac{x}{3t}-\mu$$

where $\mu$ is a constant. Determine the equations of the other family of characteristics.

## 2.4.3. Reduction to canonical forms

We shall now consider equations of the type

$$Rr+Ss+Tt+f(x,y,z,p,q)=0 \qquad (2.4.3.1)$$

which may be written in the form

$$L(z) + f(x, y, z, p, q) = 0 \qquad (2.4.3.2)$$

where L is the differential operator defined by the equation

$$L = R\frac{\partial^2}{\partial x^2} + S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2} \qquad (2.4.3.3)$$

in which R,S,T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary. By a suitable change of the independent variables, we shall show that any equation of the type (2.4.2.2) can be reduced to one of three canonical forms. Suppose we change the independent variables from x, y to $\xi, \eta$ where

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)$$

and we write z(x,y) as $\zeta(\xi, \eta)$ ;then it is readily shown that equation (2.4.2.1) takes the form

$$A(\xi_x, \xi_y)\frac{\partial^2 \zeta}{\partial \xi^2} + 2B(\xi_x, \xi_y; \eta_x, \eta_y)\frac{\partial^2 \zeta}{\partial \xi \partial \eta} + A(\eta_x, \eta_y)\frac{\partial^2 \zeta}{\partial \eta^2} = F(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta) \qquad (2.4.3.4)$$

where $A(u, v) = Ru^2 + Suv + Tv^2$ (2.4.3.5)

$$B(u_1, v_1; u_2, v_2) = Ru_1u_2 + \frac{1}{2}S(u_1v_2 + u_2v_1) + Tv_1v_2 \qquad (2.4.3.6)$$

and the function F is readily derived from the given function $f$ .

The problem now is to determine $\xi$ and $\eta$ so that equation (2.4.3.4) takes the simplest possible form. The procedure is simple when the discriminant $S^2 - 4RT$ of the quadratic form (2.4.3.5) is everywhere either positive, negative, or zero, and we shall discuss these three cases separately.

Case (a): $S^2 - 4RT > 0$ . When this condition is satisfied, the roots $\lambda_1, \lambda_2$ of the equation

$$R\alpha^2 + S\alpha + T = 0$$

are real and distinct, and the coefficients of $\partial^2\zeta/\partial\xi^2$ and $\partial^2\zeta/\partial\eta^2$ in equation (2.4.3.4) will vanish if we choose $\xi$ and $\eta$ such that

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y}, \qquad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}$$

Let us choose that

$$\xi = f_1(x, y), \qquad \eta = f_2(x, y) \qquad (2.4.3.8)$$

where $f_1 = c_1, f_2 = c_2$ are the solutions of the first-order ordinary differential equations

$$\frac{dy}{dx} + \lambda_1(x, y) = 0, \qquad \frac{dy}{dx} + \lambda_2(x, y) = 0 \qquad (2.4.3.9)$$

respectively.

Now it is easily shown that, in general,

$$A(\xi_x, \xi_y)A(\eta_x, \eta_y) - B^2(\xi_x, \xi_y; \eta_x, \eta_y) = (4RT - S^2)(\xi_x \eta_y - \xi_y \eta_x)^2 \qquad (2.4.3.10)$$

so that when the A's are zero

$$B^2 = (S^2 - 4RT)(\xi_x \eta_y - \xi_y \eta_x)^2$$

and since $S^2 - 4RT > 0$, it follows that $B^2 > 0$ and therefore that we may divide both sides of the equation by it. Hence if we make the substitutions defined by the equation (2.4.3.8) and (2.4.3.9), we find that equation (2.4.3.1) reduced to the form

$$\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = \phi(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta) \qquad (2.4.3.11)$$

**Example 1.**

Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$$

to canonical form.

In this case $R = 1, S = 0, T = -x^2$, so that the roots of the equation (2.4.3.7) are $\pm x$ and the equations (2.4.3.9) are

$$\frac{dy}{dx} \pm x = 0$$

so that we may take $\xi = y + \frac{1}{2}x^2, \eta = y - \frac{1}{2}x^2$. It is then readily verified that the equation takes the canonical form

$$\frac{\partial^2 \zeta}{\partial \xi \partial \eta} = \frac{1}{4(\xi - \eta)}\left(\frac{\partial \zeta}{\partial \xi} - \frac{\partial \zeta}{\partial \eta}\right)$$

Case (b): $S^2 - 4RT = 0$. In such circumstances the roots of equation (2.4.3.7) are equal. We define the function $\xi$ precisely as in case (a) and take $\eta$ to be any function of x, y which is independent of $\xi$. We then have, as before, $A(\xi_x, \xi_y) = 0$, and hence, from equation (2.4.3.10), $B(\xi_x \xi_y; \eta_x, \eta_y) = 0$. On the other hand, $A(\eta_x, \eta_y) \neq 0$; otherwise $\eta$ would be a function of $\xi$. Putting $A(\xi_x, \xi_y)$ and B equal to zero and dividing by $A(\eta_x, \eta_y)$, we see that the canonical form of equation (1) is, in this case,

$$\frac{\partial^2 \zeta}{\partial \eta^2} = \phi(\xi, \eta, \zeta, \zeta_\xi, \zeta_\eta) \qquad (2.4.3.12)$$

**Example 2.**

Reduce the equation

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

to canonical form.

In this example $R = 1, S = 2, T = 1$, so that it is case (b), with

$$1 + 2\alpha + \alpha^2 = 0$$

in place of equation (2.4.3.7). We thus have $\lambda_1 = -1$, so that we may take $\xi = x - y,\ \eta = x + y$. We then find that the equation reduces to the canonical form

$$\frac{\partial^2 \zeta}{\partial \eta^2} = 0$$

which is readily shown to have solution

$$\zeta = \eta f_1(\xi) + f_2(\xi)$$

where the function $f_1, f_2$ are arbitrary. Hence the original equation has solution

$$z = (x + y) f_1(x - y) + f_2(x - y)$$

Case ( c): $S^2 - 4RT < 0$. This is formally the same as case (a) except that now the roots of equation (2.4.3.7) are complex. If we go through the procedure outlined in case (a), we find that the equation (2.4.3.1) reduces to the form (2.4.3.11) but that the variables $\xi,\eta$ are not real but are in fact complex conjugates. To get a real canonical form we make the further transformation

$$\alpha=\frac{1}{2}(\xi+\eta),\qquad \beta=\frac{1}{2}(\eta-\xi)$$

and it is readily shown that

$$\frac{\partial^2\zeta}{\partial\xi\partial\eta}=\frac{1}{4}(\frac{\partial^2\zeta}{\partial\alpha^2}+\frac{\partial^2\zeta}{\partial\beta^2})$$

so that the desired canonical form is

$$\frac{\partial^2\zeta}{\partial\alpha^2}+\frac{\partial^2\zeta}{\partial\beta^2}=\psi(\alpha,\beta,\zeta,\zeta_\alpha,\zeta_\beta)$$

(2..4.3.13)

## 2.4.3.1.Self Assessment Questions:

1. Reduce the equation

$$(n-1)^2\frac{\partial^2 z}{\partial x^2}-y^{2n}\frac{\partial^2 z}{\partial y^2}=ny^{2n-1}\frac{\partial z}{\partial y}$$

to canonical form, and find its general solution.

2. Reduce the equation

$$y^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+x^2\frac{\partial^2 z}{\partial y^2}=\frac{y^2}{x}\frac{\partial z}{\partial x}+\frac{x^2}{y}\frac{\partial z}{\partial y}$$

to canonical form, and hence solve it.

## 2.4.4.Separation of Variables

Many of the second order linear equations are solvable by the separation of the independent variables. The method is based on the assumption that the substitution

$$z=X(x)Y(y) \qquad \ldots (2.4.4.1)$$

reduces the partial differential equation

$$Rr+Ss+Tt+Pp+Qq+Zz=F \qquad \ldots (2.4.4.2)$$

into an equation of the form

$$\left(\frac{1}{X}\right)f(D)X = \left(\frac{1}{Y}\right)g(D')Y \qquad \ldots (2.4.4.3)$$

where $f(D), g(D')$ are quadratic functions of $D$ and $D'$ respectively.

Since x and y are independent, such a relation is possible only when each of the expressions equals a constant. Thus, the problem of finding solution to equations of the form (2.4.4.1) reduces to solving the following pair of second order linear ordinary differential equations.

… (2.4.4.4)

$f(D)X = \lambda X$ and $g(D')Y = \lambda Y$

The principle is readily extendible to more than two independent variables.

The method is best illustrated by means of a [particular example.

Consider the one-dimensional diffusion equation

… (2.4.4.5)

$$\frac{\partial^2 z}{dx^2} = \frac{1}{k}\frac{\partial z}{\partial t}$$

If we write

$$z = X(x)T(t)$$

we find that

$$\frac{1}{x}\frac{d^2 x}{dx^2} = \frac{1}{kT}\frac{dT}{dt}$$

so that the pair of ordinary equations corresponding to (2.4.4.4) is

$$\frac{d^2 X}{dx^2} = \lambda X, \frac{dT}{dt} = k\lambda T$$

so that if we are looking for a solution which tends to zero as $t \to \infty$,

we may take

$$X = A\sum_{i=1}^{n}(X_i - \overline{X})^2 \cos(nx+\varepsilon), T = Be^{-kn^2 t}$$

where we have written $-n^2$ for $\lambda$. Thus

$$z(x,t) = \sum_{n=0}^{\infty} C_n \cos(nx+\varepsilon_n)e^{-n^2 kt} \qquad \ldots (2.4.4.6)$$

are, formally at least, solutions of equation (2.4.4.5). It should be noted that the solutions(6) have the property that $z \to 0$ as $t \to \infty$ and that

$$z(x,0) = \sum_{n=0}^{\infty} C_n \cos(nx + \varepsilon_n) \qquad \dots (2.4.4.7)$$

The principle can readily be extended to a larger number of variables. For example, if we wish to find solutions of the form

$$z = X(x)Y(y)T(t) \qquad \dots (2.4.4.8)$$

of the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{k}\frac{\partial z}{\partial t} \qquad \dots (2.4.4.9)$$

we note that for such a solution equation (2.4.4.9) can be written as

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = \frac{1}{kT}\frac{dT}{dt}$$

so that we may take

$$\frac{dT}{dt} = -n^2 kT, \frac{d^2X}{dx^2} = -l^2 X, \frac{d^2Y}{dy^2} = -m^2 Y$$

provided that

$$l^2 + m^2 = n^2$$

Hence we have solutions of equation (2.4.4.9) of the form

$$z(x,y,t) = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} C_{lm} \cos(lx + \varepsilon_l)\cos(my + \varepsilon_m)e^{-k(l^2+m^2)t} \qquad \dots (2.4.4.10)$$

**Example**: Solve

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{k}\frac{\partial z}{\partial t}$$

Solution: If we put $z = X(x)T(t)$, we note that the given equation is reduced to the form

$$\frac{1}{x}\frac{\partial^2 X}{\partial x^2}=\frac{1}{kT}\frac{dT}{dl}$$

Equating them to the constant $\lambda$, we thus have the subsidiary equations as

$$\frac{d^2 X}{dx^2}=\lambda X, \frac{dT}{dl}=k\lambda T$$

Solving the above ordinary differential equations we get

$X = A\cos(nx+\varepsilon)$ and $T = Be^{-kn^2t}$ writing $\lambda=-n^2$

Hence a solution of the given partial differential equation is $C_n\cos(nx+\varepsilon_n)e^{-kn^2t}$ where $C_n$ and $\varepsilon_n$ are constants. Since this is a solution for all values of n, we have the general solution as

$$z=\sum_{n=0}^{\alpha} C_n\cos(nx+\varepsilon_n)e^{-n^2kt}$$

Note $z\to 0$ as $t\to\alpha$ and $z_0=\sum_{n=0}^{\alpha} C_n\cos(nx+\varepsilon_n)$ where $z_0$ is the value of z when t=0.

## 2.4.4.1.Self Assessment Questions:

1. By separating the variables, show that the one-dimensional wave equation

$$\frac{\partial^2 z}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

has solutions of the form $A\exp(\pm inx \pm inct)$, where $A$ and n are constants. Hence show that functions of the form

$$z(x,t)=\sum_r \{A_r\cos\theta\frac{r\pi ct}{a}+B_r\sin\theta\frac{r\pi ct}{a}\}\sin\frac{r\pi x}{a}$$

Where $A_r$'s and $B_r$'s are constants, satisfy the wave equation and the boundary conditions z(0,t)=0,z(a,t)=0 for all t.

2. By separating the variables, show that the equation $\nabla_1^2 V=0$ has solutions of the form $A\exp(\pm nx \pm iny)$, where A and n are constants. Deduce that functions of the form

$$V(x,y)=\sum_r A_r e^{-r\pi x/a}\sin\frac{r\pi y}{a} \qquad x\geq 0, 0\leq y\leq a$$

where the $A_r's$ are constants, are plane harmonic functions satisfying the conditions $V(x,0)=0, V(x,a)=0, V(x,y)\to 0$ as $x\to\infty$.

## 2.4.5. Monge's method

Gaspard Monge (1746-1818), the inventor of descriptive geometry, has given a method of solving a second order linear partial differential equation of the form Rr+Ss+Tt=V

Where $r=\frac{\partial^2 z}{\partial x^2}, s=\frac{\partial^2 z}{\partial x\partial y}, t=\frac{\partial^2 z}{\partial y^2}$

And R, S, T and V are functions of x, y, z, p and q. Obviously, this method is more general than the methods already discussed in solving homogeneous and non-homogonous linear equations.

First note that

dp=rdx+sdy or r=(dp-sdy)/dx

and dq=sdx+tdy or t=(dq-sdx)/dy

Hence the equation Rr+Ss+Tt=V reduces to the form

$$(Rdpdy + Tdqdx - Vdxdy) - s$$
$$(Rdy^2 - Sdx\,dy + Tdx^2)=0 \qquad (2.4.5.1)$$

So, any relation between x, y, z, p and q which satisfies $Rdpdy + Tdqdx - Vdx\,dy=0$ and $Rdy^2-Sdxdy+Tdx^2=0$ also satisfies (2.4.5.1).

Now from $Rdpdy + Tdqdx - Vdxdy=0$ (2.4.5.1)

And $Rdy^2-Sdxdy+Tdx^2=0$ (2.4.5.2)

Known as Monge's subsidiary equations, together with dz=pdx+qdy, it is possible under certain conditions to derive one or two relations between x, y, z, p and q called intermediate integrals,. These intermediate integrals when solved for p and q help find out the complete integral by substituting the relevant expressions for p and q in the equation

Dz=pdx+qdy

and then integrating.

So, the rules may be stated as follows

**Step 1.** Form the equation

$$Rdy^2-Sdxdy+Tdx^2=0$$

and resolve it to the linear factors as

$(dy-m_1dx)(dy-m_2dx)=0$ where

$$m_1+m_2=\frac{S}{R},\ m_1m_2=\frac{T}{R}$$

**Step 2**. Take each of the factors $dy-m_1dx=0$ and $dy-m_2dx=0$ with $Rdpdy+Tdqdx-Vdxdy=0$ and if necessary also $dz=pdx+qdy$ to obtain two intermediate integrals $u_1=f_1(v_1)$ and $u_2=f(v_2)$ where and are two arbitrary functions.

Step 3. If $m_1=m_2$ either of the intermediate integrals may be integrated and if $m_1 \neq m_2$ both the intermediate integrals are solved for p and q and their values substituted in $dz=pdx+qdy$.

The integration of this last expression gives the complete integral.

The following examples illustrate the method.

**Example**: Solve $r+(a+b)s+abt=xy$

Solution: Comparing the given equation with $Rr+Ss+Tt=V$

We get $R=1$; $S=a+b$; $t=ab$, $V=xy$

Hence Monge's subsidiary equations are

$dy^2-(a+b)dxdy+abdx^2=0$ (i)

and $dpdy+abdpdx-xydxdy=0$ (ii)

Factorising (i) we get $(dy-adx)(dy-bdx)=0$

Now $dy-adx=0$ gives $y-ax=c_1$ (iii)

and $dy-bdx=0$ gives $y-bx=c_2$ (iv)

From (iii) and (ii) we get

$dp+bdq-x(c_1+ax)dx=0$

Or $dp+bdq-x(c_1+ax)dx=0$

Integrating we get

$$p+bq-\left[\frac{c_1}{2}x^2+\frac{a}{3}x^3\right] = \text{constant}$$

Or $$p+bq-\frac{1}{2}x^2(y-ax)-\frac{a}{3}x^3 = \text{constant} \qquad [\because y-ax=c_1]$$

So the first intermediate integral is

$$p+bq-\frac{1}{2}x^2(y-ax)-\frac{ax^3}{3} = f_1(y-ax) \cdot \qquad [\because y-ax=c_1]$$

Where $f_1$ is an arbitrary function.

The second intermediate integral is similarly obtained as

$$p+aq-\frac{1}{2}x^2(y-bx)-\frac{bx^3}{3} = f_2(y-bx)$$

Where $f_2$ is an arbitrary function

Solving the above two equations we get

$$p=\frac{1}{2}x^2y-\frac{1}{6}(a+b)x^3-\frac{1}{b-a}\{af_1(y-ax)-bf_2(y-bx)\}$$

and $$q=\frac{1}{6}x^3+\frac{1}{b-a}\{f_1(y-ax)-f_2(y-bx)\}$$

Putting these values in dz=pdx+qdy we get

$$dz=\frac{1}{2}x^2ydx+\frac{1}{6}x^3dy-\frac{(a+b)}{6}x^3dx+\frac{1}{b-a}$$
$$\{(dy-adx)f_1(y-ax)-(dy-bdx)f_2(y-bx)\}$$

Integrating, we get

$$z=\frac{1}{6}x^3y-(a+b)\frac{x^4}{24}+F_1(y-ax)+F_2(y-bx)$$

which is the complete integral of the given equation.

## 2.4.5.1.Self Assessment Questions:

1. Solve using Monge's Methods
   q (1+q)r-(p+q+2pq)s+p(1+p)t=0

2. Solve using Monge's Method
   (x-y)(xr-xs-ys+yt)=(x+y)(p-q)

3. Solve using Monge's Method
   $rx^2$-3sxy+$2ty^2$+px+2qy=x+2y

## 2.5 Let us sum up

In this unit we have covered the following:

1. We introduced the concept of partial differential equation, its occurrence and origin. Formation of partial differential equation are also discussed.
2. We discussed integral surface and orthogonal surface
3. We explained the solution of first order non-linear equations using Charpit's and Jacobi's method.
4. Solution of linear second order equation with constant co-efficient have also explained.
5. We have also explained the procedure for reduction of second order equation to canonical form
6. Finally we have explained the solution of non-linear second order equation using Monge's method.

# BLOCK - 3
# LAPLACE'S EQUATION, WAVE EQUATION AND DIFFUSION EQUATION

**Structure**



## 3.0 Introduction

In theoretical physics and engineering partial differential equations generally arise from mathematical formulation of the actual physical problem

## 3.1. Objectives

After reading this **block** you will be able to :

- Know, how Laplace's equation, wave equation and Diffusion equation appeared from real physical problem.
- Solution of these equation by separation of variables
- The theory of Green's function in Laplace's equation.

## 3. 2. Occurrence of Laplace's equation

We shall summarize here the main relations in some of the branches of physics in which the field equations can be reduced to Laplace's equation.

(a) Gravitation.

(i) Both inside and outside the attracting matter the force of attraction F can be expressed in terms of a gravitational potential $\psi$ by the equation

$$F = \text{grad } \psi$$

(ii) In empty space $\psi$ satisfies Laplace's equation $\nabla^2\psi = 0$

(iii) At any point at which the density of gravitating matter is $\rho$ the potential $\psi$ satisfies Poisson's equation $\nabla^2\psi = -4\pi\rho$

(iv) When there is matter distributed over a surface, the potential function $\psi$ assumes different forms $\psi_1, \psi_2$ on opposite sides of the surface and on the surface these two functions satisfy the conditions

$$\psi_1 = \psi_2, \qquad \frac{\partial \psi_2}{\partial n} - \frac{\partial \psi_1}{\partial n} = -4\pi\sigma$$

where $\sigma$ is the surface density of the matter and n is the normal to the surface pointing from the region 1 into the region 2.

(v) There can be no singularities in $\psi$ except at isolated masses.

(b) Irrotational Motion of a Perfect Fluid:

(i) The velocity q of a perfect fluid in irrotational motion can be expressed in terms of a velocity potential $\psi$ by the equation
$q = -grad\,\psi$

(ii) At all points of the fluid where there are no sources or sinks the function $\psi$ satisfies Laplace's equation
$\nabla^2\psi = 0$

(iii) When the fluid is in contact with a rigid surface which is moving so that a typical point P of it has velocity U, then (q-U).n=0, where n is the direction of the normal at P. The condition satisfied by $\psi$ is therefore that
$$\frac{\partial \psi}{\partial n} = -(U.n)$$
at all points of the surface.

(iv) If the fluid is at rest at infinity, $\psi \to 0$, but if there is a uniform velocity V in the z direction, this condition is replaced by the condition $\psi \approx -Vz$ as $z \to \alpha$.

(v) The function $\psi$ has no singularities except at sources or sinks.

## 3. 3. Boundary value Problems

In this chapter we have seen that in the discussion of certain physical problems the function $\varphi$ whose analytical form we are seeking must, in addition to satisfying Laplace's equation within a certain region of space V, also satisfy certain conditions on the boundary S of this region. Any problem in which we are required to find such a function $\varphi$ is called a boundary value problem for Laplace's equation.

There are two main types of boundary value problem for Laplace's equation, associated with the names of Dirichlet and Newmann. By the interior Dirichlet problem we mean the following problem:

If f is a continuous function prescribed on the boundary S of some finite region V, determine a function $\varphi(x,t,z)$ such that $\nabla^2\varphi = 0$ within V and $\varphi = f$ on S.

For instance, the problem of finding the distribution of temperature within a body in the steady state when each point of its surface is kept at a prescribed steady temperature is an interior Dirichlet problem, while that of determining the distribution of potential outside a body whose surface potential is prescribed in an exterior Dirichlet problem, while surface potential is prescribed in an exterior Dirichlet problem.

The existence of the solution of a Dirichlet problem under very general conditions can be established. Assuming the existence of the solution of an interior Dirichlet problem, it is a simple matter to prove its uniqueness. Suppose that $\psi_1$ and $\psi_2$ are both solutions of the interior Dirchilets problem in question. Then the function

$$\psi = \psi_1 - \psi_2$$

must be such that $\nabla^2\psi = 0$ within V and $\psi = 0$ on S. We know that the values of $\psi$ within V cannot exceed its maximum on S or be less than its minimum on S, so that we must have $\psi = 0$ within V; i.e., $\psi_1 \equiv \psi_2$ within V. It should also be observed that the solution of a Dirchlet problem depends continuously on the boundary function.

On the other hand the solution of the exterior Dirichlet problem is not unique unless some restriction is placed on the behaviour of $\psi(x,y,z)$ as $r \to \alpha$. In the three-dimensional case it can be proved that the solution of the exterior Dirchlet problem is unique provided that

$$|\psi(x,y,z)| < \frac{C}{r}$$

where C is a constant. In the two-dimensional case we require the function $\psi$ to be bounded at infinity.

In case where the region V is bounded the solution of the exterior Dirichlet problem can be deduced from that of a corresponding interior Dirichlet problem. Within the region V we choose a spherical surface C with center O and radius a. We next invert the space outside the region V with respect to the sphere C; i.e., we map a point P outside V into a point $\Pi$ inside the sphere C such that

$$OP.O\Pi = a^2$$

In this way the region exterior to the surface S is mapped into a region V*lying entirely within the sphere C. It can be easily shown that if

$$f^*(\Pi) = \frac{a}{O\Pi} f(P)$$

and if $\psi^*(\Pi)$ is the solution of the interior Dirchlet problem

$$\nabla^2\psi = 0 \text{ outside V}, \quad \psi = f(P) \text{ for } P \in S$$

## 3. 4. Solution of Laplace's equation

If we take the function $\psi$ to be given by the equation

$$\psi = \frac{q}{|r-r'|} = \frac{q}{\sqrt{((x-x')^2+(y-y')^2+(z-z')^2}} \tag{3.4.1}$$

where q is a constant and $(x', y', z')$ are the coordinates of a fixed point, then since

$$\frac{\partial\psi}{\partial x} = -\frac{q(x-x')}{|r-r'|^3}, etc.$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{q}{|r-r'|^3} + \frac{3q(x-x')^2}{|r-r'|^5}, etc.$$

it follows that

$$\nabla^2\psi = 0$$

showing that the function (3.4.1) is a solution of Laplace's equation except possibly at the point $(x', y', z')$, where it is not defined.

The function $\psi$ given by equation (3.4.1) is a possible form for the electrostatic potential corresponding to a space which, apart from the isolated point $(x', y', z')$, is empty of electric charge. To find the charge at this singular point we make use of Gauss' theorem. If S is any sphere with center $(x', y', z')$ then it is easily shown that

$$\int_S \frac{\partial\psi}{\partial n} dS = -4\pi q$$

from which it follows, by Gauss' theorem, that equation (3.4.1) gives the solution of Laplace's equation corresponding to an electric charge +q.
By a simple superposition procedure it follows immediately that

$$\psi = \sum_{i=1}^{n} \frac{q}{|r - r_i|} \tag{3.4.2}$$

is the solution of Laplace's equation corresponding to n charges qi situated at points with position vectors $r_i$ (i=1, 2, ...,n).
In electrical problems we encounter the situation where two charges +q and –q are situated very close together, say at points r' and $r' + \delta r'$ where $\delta r' = (l, m, n)a$. The solution of Laplace's equation corresponding to this distribution of charge is

$$\psi = \frac{q}{|r - r'|} + \frac{q}{|r - r' - \delta r'|}$$

Now

$$\frac{1}{|r - r' - \delta r'|} = \frac{1}{|r - r'|} + \frac{l(x - x') + m(y - y') + n(z - z')}{|r - r'|^3} \tag{3.4.3}$$

a result which may be written in other ways: If we introduce a vector $m = \mu(l, m, n)$, then

$$\psi = \frac{m.(r - r')}{|r - r'|^3}, etc. \tag{3.4.4}$$

Also since

$$\frac{\partial}{\partial x'} \frac{1}{|r - r'|} = \frac{x - x'}{|r - r'|^3}, etc.$$

it follows that (3.4.3) may be written in the form

$$\psi = (m.grad') \frac{1}{|r - r'|} = \mu \left( l \frac{\partial}{\partial x'} + m \frac{\partial}{\partial y'} + n \frac{\partial}{\partial z'} \right) \frac{1}{|r - r'|} \tag{3.4.5}$$

In reality we usually have to deal with continuous distributions of charge rather than with point charges or dipoles. By analogy with equation (3.4.2) we should therefore expect that when a continuous distribution of charge fills a region V of space, the corresponding form of the function $\psi$ given as

$$\psi = \int_V \frac{dq}{|r - r'|} \tag{3.4.6}$$

where q is the Stieltjes measure of the charge at the point r' or if $\rho$ denotes the charge density by

$$\psi(r) = \int_V \frac{\rho(r') d\tau'}{|r - r'|} \tag{3.4.7}$$

By a similar argument it can be shown that the solution corresponding to a surface S carrying an electric charge of density $\sigma$ is

$$\psi(r) = \int_S \frac{\sigma(r')dS'}{|r - r'|} \tag{3.4.8}$$

## 3. 5. Theory of Green's function for Laplace's equation.

We now consider of the interior Dirichlet problem. Suppose, in the first instance, that the values of $\psi$ and $\partial\psi/\partial n$ are known at every point of the boundary S of a finite region V and that $\nabla^2\psi = 0$ within V. We can determine $\psi$ by a simple application of Green's theorem in the form

$$\int_\Omega (\psi\nabla^2\psi' - \psi'\nabla^2\psi)d\tau = \int_\Sigma \left(\psi\frac{\partial\psi'}{\partial n} - \psi'\frac{\partial\psi}{\partial n}\right)dS \tag{3.5.1}$$

where $\Sigma$ denotes the boundary of the region $\Omega$

If we are interested in determining the solution $\psi(r)$ of our problem at a point P with position vector r, then we surround p by a sphere C which has its center at P and has radius $\varepsilon$ and take $\Sigma$ to be the region which is exterior to C and interior to S. Putting

$$\psi' = \frac{1}{|r' - r|}$$

and noting that

$$\nabla^2\psi' = \nabla^2\psi = 0$$

within $\Omega$ we see that

$$\int_C \left\{\psi(r')\frac{\partial}{\partial n}\frac{1}{|r'-r|} - \frac{1}{|r'-r|}\frac{\partial\psi}{\partial n}\right\}dS' +$$
$$\int \left\{\psi(r')\frac{\partial}{\partial n}\frac{1}{|r'-r|} - \frac{1}{|r'-r|}\frac{\partial\xi}{\partial n}\right\}dS' = 0 \tag{3.5.2}$$

where the normal n are in the directions. Now, on the surface of the sphere C.

$$\frac{1}{|r'-r|} = \frac{1}{\varepsilon}, \quad \frac{\partial}{\partial n}\frac{1}{|r'-r|} = \frac{1}{\varepsilon^2}$$
$$dS' = \varepsilon^2 \sin\theta d\theta d\phi$$

and

$$\psi(r') = \psi(r) + \varepsilon\left\{\sin\theta\cos\phi\frac{\partial\psi}{\partial x} + \sin\theta\sin\phi\frac{\partial\psi}{\partial y} + \cos\theta\frac{\partial\psi}{\partial z}\right\}$$

$$\frac{\partial \psi}{\partial n} = \left(\frac{\partial \psi}{\partial n}\right)_P + O(\varepsilon)$$

so that $\int\limits_C \psi(r') \frac{\partial}{\partial n} \frac{1}{|r' - r|} dS' = 4\pi\psi(r) + O(\varepsilon)$

and $\quad \int\limits_C \frac{1}{|r' - r|} \frac{\partial \psi}{\partial n} dS' = O(\varepsilon)$

Substituting these results into equation (3.5.2) and letting $\varepsilon$ tend to zero, we find that

$$\psi(r) = \frac{1}{4\pi} \int\limits_S \left\{ \frac{1}{|r' - r|} \frac{\partial \psi(r')}{\partial n} - \psi(r') \frac{\partial}{\partial n} \frac{1}{|r' - r|} \right\} dS' \qquad (3.5.3)$$

so that the value of $\psi$ at an interior point of the region V can be determined in terms of the values of $\psi$ and $\partial\psi / \partial n$ on the boundary S.
The solution of the Dirichlet problem is thus reduced to the determination of the Green's function G(r,r').

## 3. 6. Wave equation

In this chapter we shall consider the wave equation

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$$

which is a typical hyperbolic equation. This equation is sometimes written in the form

$$\nabla^2 \varphi = 0$$

where $\nabla^2$ denotes the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

If we assume a solution of the wave equation of the form

$$\psi = \Psi(x, y, z) e^{\pm ikct}$$

then the function $\psi$ must satisfy the equation

$$(\nabla^2 + k^2)\Psi = 0$$

which is called the space form of the wave equation or Helmholtz's equation.

## 3.6.1. Occurrence of Wave equation

We shall begin this chapter by listing several kinds of situations in physics which can be discussed by means of the theory of the wave equation.

(a) Transverse Vibrations of a String: If a string of uniform linear density $\rho$ is stretched to a uniform tension T, and if, in the equilibrium position, the string coincides with the x axis, then when the string is disturbed slightly from its equilibrium position, the transverse displacement y(x,t) satisfies the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad (3.6.1.1)$$

where $c^2 = T/\rho$. At any point x=a of the string which is fixed y(a,t)=0 for all values of t.

(b) Longitudinal Vibrations in a Bar. If a uniform bar of elastic material cross section whose axis coincides with Ox is stressed in such a way that each point of a typical cross section of the bar takes the same displacement $\xi(x,t)$ then

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} \qquad (3.6.1.2)$$

where $c^2 = E/\rho$, E being the Young's modulus and $\rho$ the density of the material of the bar. The stress at any point in the bar is

$$\sigma = E\frac{\partial \xi}{\partial x} \qquad (3.6.1.3)$$

For instance, suppose that the velocity of the end x=0 of the bar $0 \le x \le a$ is prescribed to be v(t), say, and that the other end x=a is free from stress. Suppose further that at that time t=0 the bar is at rest. Then the longitudinal displacement of sections of the bar are determined by the partial differential equation (3.6.1.2) and the boundary and initial conditions.

(i) $\dfrac{\partial \xi}{\partial t} = v(t) \quad \text{for} \quad x = 0$

(ii) $\dfrac{\partial \xi}{\partial x} = 0 \quad \text{for} \quad x = a$

(iii) $\xi = \dfrac{\partial \xi}{\partial t} = 0 \quad \text{at} \quad t = 0 \quad \text{for} \quad 0 \le x \le a$

(c) Longitudinal Sound Waves. If plan waves of sound are being propagated in a horn whose cross section for the section with abscissa x in A(x) in such a way that every point of that section has the same longitudinal displacement $\xi(x,t)$, then $\xi$ satisfies the partial differential equation

$$\frac{\partial}{\partial x}\left\{\frac{1}{A}\frac{\partial}{\partial x}(A\xi)\right\}=\frac{1}{c^2}\frac{\partial^2\xi}{\partial t^2} \qquad (3.6.1.4)$$

which reduces to the one-dimensional wave equation (3.6.1.2) in the case in which the cross section is uniform. In equation (3.6.1.4)

$$c^2=\left(\frac{dp}{d\rho}\right)_0 \qquad (3.6.1.5)$$

where the suffix 0 denotes that we take the value of $dp/d\rho$ in the equilibrium state. The change in pressure in the gas from the equilibrium value $\rho_0$ is given by the formula

$$p-p_0=-c^2\rho_0\frac{\partial\xi}{\partial x} \qquad (3.6.1.6)$$

where $\rho_0$ is the density of the gas in the equilibrium state. For instance, if we are considering the motion of the gas when a sound wave passes along a tube which is free at each of the ends x=0, x=a then we must determine solutions of equation (4) which are such that

$$\frac{\partial\xi}{\partial x}=0 \qquad \text{at } x=0 \text{ and at } x=a$$

(d) Electric Signals in Cables. We have already remarked that if the resistance per unit length R, and the leakage parameter G are both zero, the voltage V (x, t) and the current z (x, t) both satisfy the one-dimensional wave equation, with wave velocity c defined by the equation

$$c^2=\frac{1}{LC} \qquad (3.6.1.7)$$

where L is the inductance and C the capacity per unit length.

## 3. 6.2. Elementary Solution of one dimensional Wave equation

General solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad (3.6.2.1)$$

is

$$y=f(x+ct)+g(x-ct) \qquad (3.6.2.2)$$

where the functions f and g are arbitrary. In this section we hall show how this solution may be used to describe the motion of a string.

In the first instance we shall assume that the string is of infinite extent and that at time t=0 the displacement and the velocity of the string are both prescribed so that

$$y = \eta(x), \qquad \frac{\partial y}{\partial t} = v(x) \qquad \text{at} \quad t = 0 \qquad (3.6.2.3)$$

Our problem then is to solve equation (3.6.2.1) subject to the initial conditions (3.6.2.3). Substituting from (3.6.2.3) into (3.6.2.2) we obtain the relations

$$\eta(x) = f(x) + g(x), \qquad v(x) = cf'(x) - cg'(x) \qquad (3.6.2.4)$$

Integrating the second of these relations, we have

$$f(x) - g(x) = \frac{1}{c}\int_b^x v(\xi)\,d\xi$$

where b is arbitrary. From this equation and the first of the equations (3.6.2.4) we obtain the formulas

$$f(x) = \frac{1}{2}\eta(x) + \frac{1}{2c}\int_b^x v(\xi)\,d\xi$$

$$g(x) = \frac{1}{2}\eta(x) - \frac{1}{2c}\int_b^x v(\xi)\,d\xi$$

Substituting these expressions in equation (3.6.2.2), we obtain the solution

$$y = \frac{1}{2}\{\eta(x+ct) + \eta(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\,d\xi \qquad (3.6.2.5)$$

The solution (3.6.2.5) is known as d' Alembert's solution of the one dimensional wave equation. If the string is released from rest, $v = 0$ so that equation (3.6.2.5) becomes

$$y = \frac{1}{2}\{\eta(x+ct) + \eta(x-ct)\} \qquad (3.6.2.6)$$

showing that the subsequent displacement of the string is produced by two pulses of "shape". $y = \frac{1}{2}\eta(x)$ each moving with velocity c, one to the right and the other to the left. In such motion the initial displacement is

$$\eta(x) = \begin{cases} 0 & x < -a \\ 1 & |x| < a \\ 0 & x > a \end{cases}$$

The motion may be represented by a series of graphs corresponding to various values of t.

We shall now consider the motion of a semi infinite string $x \geq 0$ fixed at the point x=0. The conditions (3.6.2.3 are now replaced by

$$y = \eta(x), \quad \frac{\partial y}{\partial t} = v(x) \qquad x \geq 0 \quad \text{at} \quad t = 0 \quad (3.6.2.7a)$$

$$y = 0, \quad \frac{\partial y}{\partial t} = 0 \qquad t \geq 0 \quad \text{at} \quad x = 0 \qquad (3.6.2.7b)$$

The solution (3.6.2.5) is no longer applicable, since $\eta(x - ct)$ would not have a meaning if $t > x / c$. Suppose, however, we consider an infinite string subject to the initial conditions

$$y = Y(x), \quad \frac{\partial y}{\partial t} = V(x) \qquad \text{at} \quad t = 0$$

where
$$Y(x) = \begin{cases} \eta(x) & \text{if} \quad x \geq 0 \\ -\eta(-x) & \text{if} \quad x < 0 \end{cases}$$

and
$$V(x) = \begin{cases} v(x) & \text{if} \quad x \geq 0 \\ -v(-x) & \text{if} \quad x < 0 \end{cases}$$

Then its displacement is given by

$$y = \frac{1}{2}\{Y(x + ct) + Y(x - ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\,d\xi \qquad (3.6.2.8)$$

so that when x=0

$$y = \frac{1}{2}\{Y(ct) + Y(-ct)\} + \frac{1}{2c}\int_{-ct}^{ct} v(\xi)\,d\xi \qquad (3.6.2.9)$$

and
$$\frac{\partial y}{\partial t} = \frac{1}{2}c\{Y'(ct) - Y'(-ct)\} + \frac{1}{2}\{V(ct) - V(-ct)\}$$

It is obvious from the definitions of Y and V that both these functions are identically zero for all values of t and that therefore the function (3.6.2.9) satisfies the condition (3.6.2.7b) as well as the differential equation (3.6.2.1). It is easily verified that it also satisfies the condition (3.6.2.7a). In particular, if the string is released from rest so that v, and consequently V, is identically zero, we find that the appropriate solution is

$$y = \begin{cases} \frac{1}{2}[\eta(x + ct) + \eta(x - ct)] & x \geq ct \\ \frac{1}{2}[\eta(x + ct) - \eta(ct - x)] & x \leq ct \end{cases}$$

It may be obtained directly from the analytical form of the solution or, more easily, from the graphical solution for an infinite string subject to an initial displacement Y(x).

A similar procedure is applicable in the case of a finite string of length l occupying the space $0 \le x \le l$. The initial conditions may then be written in the form

$$y = \eta(x), \quad \frac{\partial y}{\partial t} = v(x) \qquad 0 \le x \le l \quad \text{at} \; . \; t = 0$$

$$y = 0, \quad \frac{\partial y}{\partial t} = 0 \qquad t \ge 0 \quad \text{at} \quad x = 0 \quad \text{and} \quad x = l$$

and by a method similar to the one above it is readily shown that the solution of the wae equation (3.6.2.1) satisfying these conditions is the expression (3.6.2.8), where now the function Y(x) is defined by the relations

$$Y(x) = \begin{cases} \eta(x) & \text{if} \quad 0 \le x \le l \\ -\eta(-x) & \text{if} \quad -l \le x < 0 \end{cases}$$

It is well known from the theory of Fourier series that such an odd periodic function has a Fourier sine expansion of the form

$$Y(x) = \sum_{m=0}^{\infty} \eta_m \sin\frac{m\pi x}{l} \qquad (3.6.2.10)$$

where the coefficients $\eta_m$ are given by the formula

$$\eta_m = \frac{2}{l}\int_0^l \eta(\xi)\sin\left(\frac{m\pi \xi}{l}\right)d\xi \qquad (3.6.2.11)$$

Similarly

$$V(x) = \sum_{m=0}^{\infty} v_m \sin\left(\frac{m\pi x}{l}\right) \qquad (3.6.2.12)$$

where $v_m = \frac{2}{l}\int_0^l v(\xi)\sin\left(\frac{m\pi \xi}{l}\right)d\xi \qquad (3.6.2.13)$

Substituting the results

$$\frac{1}{2}\{Y(x+ct) + Y(x-ct)\} = \sum_{m=0}^{\infty} \eta_m \sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)$$

$$\frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\,d\xi = \frac{1}{\pi c}\sum_{m=0}^{\infty}\frac{v_m}{m}\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)$$

which follow from these expansions, into the solution (3.6.2.8) we find that the solution of the present problem is

$$y = \sum_{m=0}^{\infty} \eta_m \sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{m\pi ct}{l}\right) + \frac{1}{\pi c}\sum_{m=0}^{\infty}\frac{v_m}{m}\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{m\pi ct}{l}\right)$$

---- (3.6.2.14)

Where $\eta_m$ and $\nu_m$ are defined by equations (3.6.2.11) and (3.6.2.13) respectively.

**Example**. The points of trisection of a string are pulled aside through a distance $\varepsilon$ on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string is released from rest. Show that the mid-point of the string always remains at rest.

In this case we may take l=3a and

$$\eta(x) = \begin{cases} \dfrac{\varepsilon x}{a} & 0 \le x \le a \\ \varepsilon \dfrac{(3a - 2x)}{a} & a \le x \le 2a \\ \varepsilon \dfrac{(x - 3a)}{a} & 2a \le x \le 3a \end{cases}$$

and $\nu(x) = 0$. Thus the Fourier coefficnets are

$$\eta_m = \frac{2\varepsilon}{3a^2}\left\{\int_0^a x \sin\frac{m\pi x}{3a}dx + \int_a^{2a}(3a - 2x)\sin\frac{m\pi x}{3a}dx + \int_{2a}^{3a}(x - 3a)\sin\frac{m\pi x}{3a}dx + \right\}$$

$$= \frac{18\varepsilon}{\pi^2 m^2}\left\{1 + (-1)^m\right\}\sin\left(\frac{1}{3}m\pi\right)$$

and $\nu_m = 0$

so that the displacement is given by the expression

$$y = \frac{18\varepsilon}{\pi^2}\sum_{m=1}^{\infty}\frac{1 - (-1)^m}{m^2}\sin\frac{m\pi}{3}\sin\frac{m\pi x}{3a}\cos\frac{m\pi ct}{3a}$$

which is equivalent to

$$y = \frac{9\varepsilon}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{2n\pi}{3}\sin\frac{2n\pi x}{3a}\cos\frac{2n\pi ct}{3a}$$

The displacement of the mid-point of the string is obtained by putting x=3a/2 in this expression. Since $\sin(2n\pi x/3a)$ would then equal $\sin n\pi$ and this is zero for all integral values of n, we see that the displacement of the mid-point of the string is always zero.

## 3.6.2.1.Self Assessment Questions:

1. If the string is released from rest in the position

   $$y = \frac{4\varepsilon}{l^2}x(l - x)$$

   show that its motion is described by the equation

$$y=\frac{32\varepsilon}{3}\sum_{n=0}^{\infty}\frac{2}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{l}\cos\frac{(2n+1)\pi ct}{l}$$

## 3.7. Diffusion Equation

In this chapter we shall consider the typical parabolic equation

$$k\frac{\partial^2\theta}{\partial x^2}=\frac{\partial\theta}{\partial t}$$

and its generalizations to two and three dimensions. Because of its occurrence in the analysis of diffusion phenomena we shall refer to this equation as the one-dimensional diffusion equation and to its generalization

$$k\nabla^2\theta=\frac{\partial\theta}{\partial t}$$

as the diffusion equation.

### 3.7.1. Occurrence of Diffusion equation

Here we will discuss the occurrence of diffusion equations in theoretical physics.

(a) The Conduction of Heat in Solids. If we denote by $\theta$ the temperature at a point in a homogeneous isotropic solid, then it is readily shown that the rate of flow of heat per unit area across any point plane is

$$q=-k\frac{\partial\theta}{\partial n} \qquad (3.7.1.1)$$

where k is the thermal conductivity of the solid and the operator $\partial/\partial n$ denotes differentiation along the normal. Considering the flow of heat through a small element of volume, we show that the variation of $\theta$ is governed by the equation

$$\rho\, c\frac{\partial\theta}{\partial t}=\operatorname{div}(k\operatorname{grad}\theta)+H(r,\theta,t) \qquad (3.7.1.2)$$

where $\rho$ is the density and c the specific heat of the solid and $H(r,\theta,t)d\tau$ is the amount of heat generated per unit time in the element $d\tau$ situated at the point with position vector $r$.

The heat function $H(r,\theta,t)$ may arise because the solid is undergoing radioactive decay or is absorbing radiation. A term of this kind exists also when there is generation or absorption of heat in the solid as a result of a chemical reaction, e.g., the hydration of cement.

If the conductivity k is a constant throughout the body and if we write

$$\kappa = \frac{k}{\rho c}, \qquad Q(r,\theta,t) = \frac{H(r,\theta,t)}{\rho c}$$

equation (3.7.1.2) reduces to the form

$$\frac{\partial \theta}{\partial t} = \kappa \nabla^2 \theta + Q(r,\theta,t) \tag{3.7.1.3}$$

The fundamental problem of the mathematical theory of the conduction of heat is the solution of equation (3.7.1.2) when it is known that the boundary surfaces of he solid are treated in a prescribed manner. The boundary conditions are usually of three main types:

(i) The temperature is prescribed all over the boundary; i.e., the temperatures $\theta(r,t)$ is a prescribed function of t for every point r of the boundary surface.

(ii) The flux of heat across the boundary is prescribed; i.e., $\partial\theta / \partial n$ is prescribed;

(iii) There is radiation from the surface into a medium of fixed temperature $\theta_0$ ; i.e.,

$$\frac{\partial \theta}{\partial n} + h(\theta - \theta_0) = 0 \tag{3.7.1.4}$$

where h is a constant.

If we introduce the differential operator

$$\lambda = C_0 + C_1 \frac{\partial}{\partial x} + C_2 \frac{\partial}{\partial y} + C_3 \frac{\partial}{\partial z} \tag{3.7.1.5}$$

where $C_0$, $C_1$, $C_2$. $C_3$ are functions of x, y, z only, we see that the general boundary condition

$$\lambda\, \theta(r,t) = G(r,t) \qquad r \in S \tag{3.7.1.6}$$

embraces all three cases.

(b) The slowing down of Neutrons in Matter. Under certain circumstances the one dimensional transport equations governing the slowing down of neutrons in matter can be reduced to the form

$$\frac{\partial \chi}{\partial \theta} = \frac{\partial^2 \chi}{\partial z^2} T(z,\theta) \tag{3.7.1.7}$$

where $\theta$ is the "symbolic age" and $\gamma(z,\theta)$ is the number of neutrons per unit time which reach the age $\theta$; i.e., $\gamma$ is the slowing down density. The function T is related to S(z,u) the number of neutrons being produced per unit time and per unit volume, by the relation

$$T(z,\theta) = 4\pi\, S(z,u) \frac{du}{d\theta} \tag{3.7.1.8}$$

where u=log($E_0$/E) is a dimensionless parameter expressing the energy E of he neutron in terms of a standard energy $E_0$.

(c) The diffusion of Vorticity. In the case of a viscous fluid of density $\rho$ and coefficient of viscosity $\mu$ which is started into motion from rest the vorticity $\xi$ which is related to the velocity q in the fluid by the equation

$$\xi = \text{curl } q \tag{3.7.1.9}$$

is governed by the diffusion equation

$$\frac{\partial \zeta}{\partial t} = \nu \nabla^2 \zeta \tag{3.7.1.10}$$

where $\nu = \mu / \rho$ is the kinematic viscosity.

## 3.7.1.1.Self Assessment Questions:

1. Suppose that the diffusion is linear with boundary conditions $c=c_1$ at $x=0$, $c=c_2$ at $x=l$ and that the diffusion coefficient D is given by a formula of the type $D=D_0[1+f(c)]$, where $D_0$ is a constant. Show that if the concentration distribution for the steady state has been measured, the function f(c) can be determined by means of the relation

$$l\,[\,c + F(c) - c_1 - F(c_1)\,] = x\,[c_2 + F(c_2) - c_1 - F(c_1)]$$

where $F(c) = \int_0^c f(u)\,du$

Show further that if s is the quantity of solute passing per unit area during time t, then

$$D_0 = \frac{sl}{t[c_1 + F(c_1) - c_2 - F(c_2)]}$$

## 3.7.2. Solution of Diffusion Equation

In this section we shall consider elementary solutions of the one-dimensional diffusion equation

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial \theta}{\partial t} \tag{3.7.2.1}$$

We begin by considering the expression

$$\theta = \frac{1}{\sqrt{t}} \exp\left(-\frac{x^2}{4\kappa t}\right) \tag{3.7.2.2}$$

For this function it is readily seen that

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{x^2}{4\kappa^2 t^{5/2}} e^{-x^2/4\kappa t} - \frac{1}{2\kappa t^{3/2}} e^{-x^2/4\kappa t}$$

and $$\frac{\partial \theta}{\partial t} = \frac{x^2}{4\kappa t^{5/2}} e^{-x^2/4\kappa t} - \frac{1}{2t^{3/2}} e^{-x^2/4\kappa t}$$

showing that the function (3.7.2.2) is a solution of the equation (3.7.2.1).

It follows immediately that

$$\frac{1}{2\sqrt{\pi \kappa t}} e^{-(x-\xi)^2/4\kappa t} \qquad (3.7.2.3)$$

where $\xi$ is an arbitrary real constant, is also a solution. Furthermore, if the function $\phi(x)$ is bounded for all real values of x, then it is possible that the integral

$$\frac{1}{2\sqrt{\pi \kappa t}} \int_{-\infty}^{\infty} \phi(\xi) \exp\left\{-\frac{(x-\xi)^2}{4\kappa t}\right\} d\xi \qquad (3.7.2.4)$$

is also, in some sense, a solution of the equation (3.7.2.1).

It may readily be proved that the integral (3.7.2.4) is convergent if t>0 and that the integrals obtained form it by differentiating under the integral sign with respect to x and t are uniformly convergent in the neighborhood of the point (x,t). The function $\theta(x,t)$ and its derivatives of all orders therefore exit for t>0 and since the integrand satisfies the one dimensional diffusion equation, it follows that $\theta(x,t)$ itself satisfies that equation for t>0

Now

$$\left| \frac{1}{2(\pi \kappa t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \phi(\xi) \exp\left\{-\frac{(x-\xi)^2}{4\kappa t}\right\} d\xi - \phi(x) \right|$$

$$= |I_1 + I_2 + I_3 + I_4|$$

where,

$$I_1 = \frac{1}{\sqrt{\pi}} \int_{-N}^{N} \left\{\phi\left(x + 2u\sqrt{\kappa t}\right) - \phi(x)\right\} e^{-u^2} du$$

$$I_2 = \frac{1}{\sqrt{\pi}} \int_{N}^{\infty} \phi\left(x + 2u\sqrt{\kappa t}\right) e^{-u^2} du$$

$$I_3 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{-N} \phi\left(x + 2u\sqrt{\kappa t}\right) e^{-u^2} du$$

$$I_4 = \frac{2\phi(x)}{\sqrt{\pi}} \int_{N}^{\infty} e^{-u^2} du$$

If the function $\phi(x)$ is bounded, we can make each of the integrals $I_2$, $I_3$, $I_4$ as small as we please by taking N to be sufficiently large and by the continuity of the function $\phi$ we can make the integral $I_1$ as small as we please by taking t sufficiently small. Thus as $t \to 0, \theta(x,t) \to \phi(x)$. Thus the Poisson integral

$$\theta(x,t) = \frac{1}{2(\pi\kappa t)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \phi(\xi) \exp\left\{-\frac{(x-\xi)^2}{4\kappa t}\right\} d\xi \tag{3.7.2.5}$$

is the solution of the initial value problem

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t} \qquad -\infty < x < \infty \tag{3.7.2.6}$$

$$\theta(x,0) = \phi(x)$$

It will be observed that by a simple change of variable we can express the solution (3.7.2.5) in the form

$$\theta(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \phi\left(x + 2u\sqrt{\kappa t}\right) e^{-u^2} du \tag{3.7.2.7}$$

We shall now show how this solution may be modified to obtain the solution of the boundary value problem

$$\frac{\partial^2\theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\theta}{\partial t} \qquad 0 \le x < \infty$$

$$\begin{aligned} \theta(x,0) &= f(x) \quad x > 0 \\ \theta(0,t) &= 0 \qquad t > 0 \end{aligned} \tag{3.7.2.8}$$

If we write

$$\phi(x) = \begin{cases} f(x) & \text{for } x > 0 \\ -f(-x) & \text{for } x < 0 \end{cases}$$

then the Poisson integral (4) assumes the form

$$\theta(x,t) = \frac{1}{2\sqrt{\pi\kappa t}} \int_0^{\infty} f(\xi)\left\{e^{-(x-\xi)^2/4\kappa t} - e^{-(x+\xi)^2/4\kappa t}\right\} d\xi \tag{3.7.2.9}$$

and it is readily verified that this is the solution of the boundary value problem (3.7.2.8). We may express the solution (3.7.2.9) in the form

$$\theta(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/2\sqrt{\kappa t}}^{\infty} f\left(x + 2u\sqrt{\kappa t}\right) e^{-u^2} du - \frac{1}{\sqrt{\pi}} \int_{x/2\sqrt{\kappa t}}^{\infty} f\left(-x + 2u\sqrt{\kappa t}\right) e^{-u^2} du \tag{3.7.2.10}$$

Thus if the initial temperature is a constant, $\theta_0$ say then

$$\theta(x,t) = \theta_0 erf\left\{\frac{x}{2\sqrt{\kappa t}}\right\} \qquad (3.7.2.11)$$

where

$$erf\ z = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-u^2}du \qquad (3.7.2.12)$$

The function

$$\theta(x,t) = \theta_0\left[1 - erf\left\{\frac{x}{2\sqrt{\kappa t}}\right\}\right] \qquad (3.7.2.13)$$

will therefore have the property that $\theta(x,0) = 0$. Furthermore $\theta(x,0) = \theta_0$. Thus the function

$$\theta(x,t,t') = g(t')\left[1 - erf\left\{\frac{x}{2\sqrt{\kappa t}}\right\}\right]$$

is the function which satisfies the one dimensional diffusion equation and the conditions $\theta(x,0,t') - 0, \theta(0,t,t') = g(t')$. By applying Duhamel's theorem it follows that the solution of the boundary value problem

$$\theta(x,0) = 0, \qquad \theta(0,t) = g(t) \qquad (3.7.2.14)$$

$$\theta(x,t) = \frac{2}{\sqrt{\pi}}\frac{\partial}{\partial t}\int_0^t g(t')dt' \int_{x/2\{\kappa(t-t')\}^{\frac{1}{2}}}^{\infty} e^{-u^2}du$$

$$= \frac{x}{2\sqrt{\pi\kappa}}\int_0^t g(t')\frac{e^{-x^2/4\kappa(t-t')}}{(t-t')^{3/2}}dt'$$

Changing the variable of integration from $t'$ to u where

$$t' = t - \frac{x^2}{4\kappa u^2}$$

we see that the solution may be written in the form

$$\theta(x,t) = \frac{2}{\sqrt{\pi}}\int_{\eta}^{\infty} g\left(t - \frac{x^2}{4\kappa u^2}\right)e^{-u^2}du, \qquad \eta = \frac{x}{2\sqrt{\kappa t}} \qquad (3.7.2.15)$$

## 3.7.2.1.Self Assessment Questions:

1. The surface x=0 of the semi-infinite solid x>0 is kept at temperature $\theta_0$ during 0<t<T and is maintained at zero temperature for t>T. Show that if t>T

$$\theta(x,t) = \theta_0 \left\{ erf \frac{x}{2\sqrt{\kappa(t-T)}} - erf \frac{x}{2\sqrt{\kappa t}} \right\}$$

and determine the value of $\theta$ if $t<T$.

## 3.8 Let us sum up:

We have discussed the following in this block.

1. We have given examples from physics how Laplace's equation, Wave equation and Diffusion equation are occurred from actual physical problem.
2. We have also discussed the solution of these equations by the method of separation of variables.

# BLOCK 4
# VOLTERRA INTEGRAL EQUATION

**Structure**



## 4.0 Introduction

The theory of integral equation is one of the most important branches of mathematics, particularly on its' importance in boundary value problems in ordinary and partial differential equations. Integral equations occurred in the field of mechanics and mathematical physics, like mechanical vibration, theory of analytic functions, orthogonal system, Quadratic forms of many variables etc. Integral equation also occur in the problems of science and technology like radiation transfer problem, neutron diffusion problem etc. A differential equation can be replaced by an integral equation with the help of initial and boundary conditions. As such, each solution of the integral equation automatically satisfies the boundary conditions.

## 4.1 Objectives

After reading this block, you will be able to :

- understand the concept of integral equation
- find the relation between a differential equation and in integral equation
- Know the concept of Volterra equation by successive approximations.
- Know the convolution type equation
- Volterra equation in the line *(x, )*
- Euler integral, Abel's problem and Abel's integral equation and its generalisation
- Volterra equation of first kind of the convolution type

## 4.2. Basic Concepts

The equation

$$\varphi(x) = f(x) + \lambda \int_a^x K(x,t)\varphi(t)dt \qquad \text{......(4.2.1)}$$

Where $f(x)$, $K(x, t)$ are known functions, $\varphi(x)$ is the unknown function and l is a numerical parameter, is called Volterra's linear integral equation of the second kind. The function $K(x, t)$ is the kernel of Volterra's equation. If $f(x) \equiv 0$, then equation (4.2.1) takes the form

$$\varphi(x) = \lambda \int_a^x K(x,t)\varphi(t)dt \qquad \text{... (4.2.2)}$$

and is called a *homogeneous* Volterra equation of the second kind.

The equation

$$\lambda \int_a^x K(x,t)\varphi(t)dt = f(x) \qquad \text{..... (4.2.3)}$$

where $\varphi(x)$ is the unknown function is called Volterra's integral equation of the first kind. Without loss of generality, we can consider the lower limit $a$ as equal to zero (in the sequel we shall assume this to be the case).

A solution of the integral equation (4.2.1), (4.2.2) or (4.2.3) is a function $\varphi(x)$, which, when substituted into the equation, reduces it to an identity (with respect to x).

**Example** Show that the function $\varphi(x)=\dfrac{1}{(1+x^2)^{3/2}}$ is a solution of the Volterra integral equation

$$\varphi(x)=\frac{1}{1+x^2}-\int_0^x \frac{1}{1+x^2}\varphi(t)dt \qquad (4.2.4)$$

Solution: Substituting the function $\dfrac{1}{(1+x^2)^{3/2}}$ in place of $\varphi(x)$ into the right member of (4), we obtain

$$\frac{1}{1+x^2}-\int_0^x \frac{1}{1+x^2}\frac{1}{(1+t^2)^{3/2}}dt=\frac{1}{1+x^2}-\frac{1}{1+x^2}\left(-\frac{1}{(1+t^2)^{3/2}}\right)\Bigg|_{t=0}^{t=x}$$

$$=\frac{1}{1+x^2}+\frac{1}{(1+x^2)^{3/2}}-\frac{1}{1+x^2}$$

$$=\frac{1}{(1+x^2)^{3/2}}=\varphi(x)$$

Thus, the substitution of $\varphi(x)=\dfrac{1}{(1+x^2)^{3/2}}$ into both sides of equation (4.2.4) reduces the equation to an identity with respect to $x$:

$$\frac{1}{(1+x^2)^{3/2}}\equiv\frac{1}{(1+x^2)^{3/2}}$$

According to the definition, this means that $\varphi(x)=\dfrac{1}{(1+x^2)^{3/2}}$ is a solution of the integral equation (4.2.4).

## 4.2.1. Self Assessment Questions:

Verify that the given functions are solutions of the corresponding integral equations

1. $\varphi(x)=\frac{x}{\left(1+x^2\right)^{5/2}}$;

$$\varphi(x)=\frac{3x+2x^3}{3\left(1+x^2\right)^2}-\int_0^x\frac{3x+2x^3-t}{\left(1+x^2\right)^2}\varphi(x)\,dt$$

2. $\varphi(x)=e^x\left(\cos e^x-e^x\sin e^x\right)$

3. $\varphi(x)=\left(1-xe^{2x}\right)\cos 1-e^{2x}\sin 1+\int_0^x\left[1-(x-t)e^{2x}\right]\varphi(t)dt$

4. $\varphi(x)=x-\frac{x^3}{6}$; $\varphi(x)=x-\int_0^x\sinh(x-t)\varphi(t)\,dt$

5. $\varphi(x)=1-x$; $\int_0^x e^{x-t}\varphi(t)\,dt=x$

Note. Volterra-type integral equations occur in problems of physics in which the independent variable varies in a preferential direction (for example, time, energy, etc.).

Consider a beam of X-rays traversing a substance in the direction of the x-axis. We will assume that the beam maintains that direction when scattered. Consider a collection of rays of specified wavelength. When passing through a thickness dx, some of the rays are absorbed; others undergo a change in wavelength due to scattering. On the other hand, the collection is augmented by those rays which, though originally of greater energy (i.e., shorter wavelength λ), lose part of their energy through scattering. Thus, if the function $f(\lambda,\ x)d\lambda$ gives the collection of rays whose wavelength lies in the interval from $\lambda$ to $\lambda+\delta\lambda$, then

$$\frac{\partial f(\lambda,x)}{\partial x}=-\mu\ f(\lambda,x)+\int_0^\lambda P(\lambda,\tau)f(\tau,x)d\tau$$

where $\mu$ is the absorption coefficient and $P(\lambda,\ \tau)d\tau$ is the probability that in passing through a layer of unit thickness a ray of wavelength $\tau$ acquires a wavelength which lies within the interval between $\lambda$ to $\lambda+\delta\lambda$.

What we have is an *integro -differential equation*, i.e., an equation in which the unknown function $f(\lambda,\ x)$ is under the sign of the derivative and the integral.

Putting

$$f(\lambda,x)=\int_0^\infty e^{-px}\psi(\lambda,p)dp$$

Where $\psi(\lambda,\ p)$ is a new unknown function, we find that $\psi(\lambda,\ p)$ will satisfy the Volterra integral equation of the second kind

$$\psi(\lambda, x) = \frac{1}{\mu - p}\int_0^\infty P(\lambda,\tau)\psi(\tau, p)d\tau$$

## 4.3. Relationship Between Linear Differential Equations and Volterra Integral Equations

The solution of the liner differential equation

$$\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_n(x)y = F(x) \qquad \ldots (4.3.1)$$

with continuous coefficients $a_i(x)$ ( $i = 1, 2, \ldots, n$), given the initial conditions

$$y(0) = C_0,\ y'(0) = C_1,\ \ldots,\ y^{(n-1)}(0) = C_{n-1} \qquad \ldots.(4.3.2)$$

may be reduced to a solution of some Volterra integral equation of the second kind.
Let us demonstrate this in the case of a differential equation of the second order

$$\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = F(x) \qquad \ldots. (4.3.1')$$

$$y(0) = C_0,\ y'(0) = C_1 \qquad \ldots.. (4.3.2')$$

Put $$\frac{d^2 y}{dx^2} = \varphi(x) \qquad \ldots.. (4.3.3)$$

Whence, taking into account the initial conditions (4.3.2$'$ ), we successively find

$$\frac{dy}{dx} = \int_0^x \varphi(t)dt + C_1, \quad y = \int_0^x (x-t)\varphi(t)dt + C_1 x + C_0 \qquad \ldots.. (4.3.4)$$

Here, we utilized the formula

$$\underbrace{\int_{x_0}^{x} dx \int_{x_0}^{x} dx \ldots \int_{x_0}^{x}}_{n} f(x)dx = \frac{1}{(n-1)!}\int_{x_0}^{x}(x-z)^{n-1} f(z)dz$$

Taking into account (4.3.3) and (4.3.4), differential equation (4.3.1) may be written as follows:

$$\varphi(x) + \int_0^x a_1(x)\varphi(t)dt + C_1 a_1(x) + \int_0^x a_2(x)(x-t)\varphi(t)dt + C_1 x a_2(x) + C_0 a_2(x) = F(x)$$

$$\text{or } \varphi(x) + \int_0^x [a_1(x) + a_2(x)(x-t)]\varphi(t)dt$$

$$= F(x) - C_1 a_1(x) - C_1 x a_2(x) - C_0 a_2(x) \quad \text{... (4.3.5)}$$

Putting

$$K(x, t) = -[a_1(x) + a_2(x)(x - t)] \quad \text{.... (4.3.6)}$$

$$f(x) = F(x) - C_1 a_1(x) - C_1 x a_2(x) - C_0 a_2(x) \quad \text{.....(4.3.7)}$$

We reduce (4.3.5) to the form

$$\varphi'(x) = \int_0^x \mathrm{K}(x,t)\varphi(t)dt + f(x) \quad \text{....}$$

(4.3.8)

which means that we arrive at a Volterra integral equation of the second kind.

The existence of a unique solution of equation (4.3.8) follows from the existence and uniqueness of solution of the Cauchy problem $(4.3.1')$ – $(4.3.2')$ for a linear differential equation with continuous coefficients in the neighbourhood of the point $x = 0$.

Conversely, solving the integral equation (4.3.8) with K and f determined form (4.3.6) and (4.3.7), and substituting the expression obtained for $\varphi(x)$ into the second equation of (4.3.4), we get a unique solution to equation $(4.31')$ which satisfies the initial condition $(4.3.2')$.

**Example**. Form an integral equation corresponding to the differential equation

$$y'' + xy' + y = 0 \quad \text{...... (1)}$$

and the initial conditions

$$y(0) = 1,\ y'(0) = 0 \quad \text{...... (2)}$$

Solution:

$$\text{Put} \quad \frac{d^2 y}{dx^2} = \varphi(x) \quad \text{...... (3)}$$

Then

$$\frac{dy}{dx} = \int_0^x \varphi(t)dt + y'(0) = \int_0^x \varphi(t)dt,\ y = \int_0^x (x-t)\varphi(t)dt + 1 \quad \text{..... (4)}$$

Substituting (3) and (4) into the given differential equation, we get

$$\varphi(x) + \int_0^x x\varphi(t)dt + \int_0^x (x-t)\varphi(t)dt + 1 = 0$$

or

$$\varphi(x) = -1 - \int_0^x (2x-t)\varphi(t)dt$$

## 4.4. Volterra Integral Equations

## 4.4.1. Resolvent kernel of Volterra Integral Equation.

Suppose we have a Volterra integral equation of the second kind

$$\varphi(x) = f(x) + \lambda \int_0^x K(x,t)\varphi(t)dt \quad (4.4.1.1)$$

Where K(x, t) is a continuous function for $0 \le x \le a$, $0 \le t \le x$, and f(x) is continuous for $0 \le x \le a$.
We shall seek the solution of integral equation (4.4.1.1) in the form of an infinite power of series λ:

$$\varphi(x) = \varphi_0(x) + \lambda \varphi_1(x) + \lambda^2 \varphi_2(x) + \dots + \lambda^n \varphi_n(x) + \dots \quad (4.4.1.2)$$

Formally substituting this series into (4.4.1.1), we obtain

$$\varphi_0(x) + \lambda \varphi_1(x) + \lambda^2 \varphi_2(x) + \dots + \lambda^n \varphi_n(x) + \dots$$

$$= f(x) + \lambda \int_0^x K(x,t)[\varphi_0(x) + \lambda \varphi_1(x) + \dots + \lambda^n \varphi_n(x) + \dots]dt \quad (4.4.1.3)$$

Comparing coefficients of like powers of λ, we find

$$\varphi_0(x) = f(x),$$

$$\varphi_1(x) = \int_0^x K(x,t)\varphi_0(t)\,dt = \int_0^x K(x,t)f(t)\,dt, \quad (4.4.1.4)$$

$$\varphi_2(x) = \int_0^x K(x,t)\varphi_1(t)\,dt = \int_0^x K(x,t)\int_0^t K(t,t_1)f(t_1)\,dt_1 dt$$

..............................................

The relations (4.4.1.4) yield a method for a successive determination of the functions $\varphi_n(x)$.It may be shown that under the assumptions made, with respect to f(x) and K(x,t), the series (4.4.1.2) thus obtained converges uniformly in x and λ for any λ and $x \in [0, a]$ and its sum is a unique solution of equation (4.4.1.1). Further, it follows from (4) that

$$\varphi_1(x) = \int_0^x K(x,t)f(t)dt \quad \dots\dots(4.4.1.5)$$

$$\varphi_2(x) = \int_0^x K(x,t)\left[\int_0^t K(t,t_1)f(t_1)dt_1\right]dt =$$

$$\int_0^x f(t_1)dt_1 \int_{t_1}^x K(x,t)K(t,t_1)dt = \int_0^x K_2(x,t_1)f(t_1)dt_1 \quad \dots\dots(4.4.1.6)$$

Where

$$K_2(x,t_1) = \int_{t_1}^{x} K(x,t)K(t,t_1)\,dt \qquad \text{...... (4.4.1.7)}$$

Similarly, it is established that, generally,

$$\varphi_n(x) = \int_0^x K_n(x,t)f(t)\,dt \quad (n = 1, 2, \ldots) \qquad \text{......}$$

(4.4.1.8)

The functions $K_n$ (x, t) are called iterated kernels. It can readily be shown that they are determined with the aid of the recursion formulas

$K_1(x, t) = K(x, t)$,

$$K_{n+1}(x,t) = \int_0^x K(x,z)K_n(z,t)\,dz \quad (n = 1, 2, \ldots) \qquad \text{...(4.4.1.9)}$$

Utilizing (4.4.1.8) and (4.4.1.9), equality (4.4.1.2) may be written as

$$\varphi(x) = f(x) + \sum_{\nu=1}^{\infty} \lambda^{\nu} \int_0^x K_{\nu}(x,t)f(t)\,dt \qquad \text{... (4.4.1.10)}$$

The function R(x, t; λ) defined by means of the series

$$R(x,t;\lambda) = \sum_{\nu=0}^{\infty} \lambda^{\nu} K_{\nu+1}(x,t) \qquad \text{... (4.4.1.11)}$$

is called the resolvent kernel (or reciprocal kernel) for the integral equation (4.4.1.1). Series (4.4.1.11) converges absolutely and uniformly in the case of a continuous kernel K(x, t).

Iterated kernels and also the resolvent kernel do not depend on the lower limit in an integral equation.

The resolvent kernel R(x, t; λ) satisfies the following functional equation:

$$R(x,t;\lambda) = K(x,t) + \lambda \int_t^x K(x,s)R(s,t;\lambda)\,ds \qquad \text{... (4.4.1.12)}$$

With the aid of the resolvent kernel, the solution of integral equation (4.4.1.1) may be written in the form

$$\varphi(x) = f(x) + \lambda \int_0^x R(s,t;\lambda)f(t)\,dt \qquad \text{... (4.4.1.13)}$$

Example: Find the resolvent kernel of the Violterra integral equation with kernel K(x, t) = 1.

Solution: We have $K_1(x, t) = K(x, t) = 1$. Further, by formulas (4.4.1.9)

$$K_2(x,t) = \int_t^x K(x,z)K_1(z,t)\,dz = \int_t^x dz = x - t,$$

$$K_4(x,t)=\int_t^x 1.\frac{(z-t)^2}{2}dz=\frac{(x-t)^3}{3!}$$

..................................

$$K_n(x,t)=\int_t^x 1.K_{n-1}(z,t)dz=\int_t^x 1.\frac{(z-t)^{n-2}}{(n-2)!}dz=\frac{(x-t)^{n-1}}{(n-1)!}$$

Thus, by eh definition of the resolvent kernel,

$$R(x,t;\lambda)=\sum_{n=0}^{\infty}\lambda^n K_{n+1}(x,t)=\sum_{n=0}^{\infty}\frac{\lambda^n(x-t)^n}{n!}=e^{\lambda(x-t)}$$

## 4.4.1.1. Self Assessment Questions:

Find the resolvent kernels for Volterra-type integral equations with the following kernels:

1. $K(x, t) = x - t$
2. $K(x, t) = e^{x-t}$
3. $K(x,t)=e^{x^2-t^2}$
4. $K(x,t)=\frac{1+x^2}{1+t^2}$
5. $K(x, t) = a^{x-t}$ $(a > 0)$

Suppose that the kernel K(x, t) is a polynomial of degree n – 1 in t so that it may be represented in the form

$$K(x,t)=a_0(x)+a_1(x)(x-t)+...+\frac{a_{n-1}(x)}{(n-1)!}(x-t)^{n-1} \qquad (4.4.1.14)$$

and the coefficients $a_k(x)$ are continuous in [0, a]. If the function g(x, t; λ) is defined as a solution of the differential equation

$$\frac{d^n g}{dx^n}-\lambda\left[a_0(x)\frac{d^{n-1}g}{dx^{n-1}}+a_1(x)\frac{d^{n-2}g}{dx^{n-2}}+...+a_{n-1}(x)g\right]=0 \qquad (4.4.1.15)$$

satisfying the condition

$$g|_{x=t}=\left.\frac{dg}{dx}\right|_{x=t}=...=\left.\frac{d^{n-2}g}{dx^{n-2}}\right|_{x=t}=0;\quad \left.\frac{d^{n-1}g}{dx^{n-1}}\right|_{x=t}=1 \qquad (4.4.1.16)$$

then the resolvent kernel R(x, t; λ) will be defined by the equality

$$R(x,t;\lambda)=\frac{1}{\lambda}\frac{d^n g(x,t;\lambda)}{dx^n} \qquad (4.4.1.17)$$

and similarly when

$$K(x,t)=b_0(t)+b_1(t)(t-x)+...+\frac{b_{n-1}(t)}{(n-1)!}(t-x)^{n-1} \qquad (4.4.1.18)$$

the resolvent kernel

$$R(x,t;\lambda)=\frac{1}{\lambda}\frac{d^n g(x,t;\lambda)}{dt^n} \qquad (4.4.1.19)$$

where g(x, t; λ) is a solution of the equation

$$\frac{d^n g}{dt^n}+\lambda\left[b_0(t)\frac{d^{n-1}g}{dt^{n-1}}+...+b_{n-1}(t)g\right]=0 \qquad (4.4.1.20)$$

which satisfies the conditions (16)

Example: Find the resolvent kernel for the integral equation

$$\varphi(x)=f(x)+\int_0^x (x-t)\varphi(t)dt$$

Solution: Here, K(x, t) = x – t; λ = 1; hence, by (14), $a_1(x) = 1$, and all the other $a_k(x) = 0$.

In this case, equation (15) has the form

$$\frac{d^2 g(x,t;1)}{dx^2}-g(x,t;1)=0$$

whence

$g(x, t; 1) = g(x, t) = C_1(t)e^x + C_2(t)e^{-x}$

conditions (16) yield

$C_1(t)e^t + C_2(t)e^{-t} = 0$

$C_1(t)e^t – C_2(t)e^{-t} = 1$ (4.4.1.21)

Solving the system (21), we find

$$C_1(t)=\frac{1}{2}e^{-t},\quad C_2(t)=-\frac{1}{2}e^{t}$$

and, consequently,

$$g(x,t)=\frac{1}{2}\left(e^{x-t}-e^{-(x-t)}\right)=\sinh(x-t)$$

According to (17)

$$R(x,t;1)=\left[\sinh(x-t)\right]_x^{\prime\prime}=\sinh(x-t)$$

## 4.4.1.2. Self Assessment Questions:

Find the resolvent kernels of integral equations with the following kernels (λ = 1):

1. K(x, t) = 2 – (x – t).
2. K(x, t) = – 2 + 3(x – t)

3. K(x, t) = 2x
4. $K(x,t) = -\frac{4x-2}{2x+1} + \frac{8(x-t)}{2x+1}$

## 4.4.2. Solution of Integral Equation by Resolvent Kernel

Suppose we have a Volterra-type integral equation, the kernel of which is dependent solely on the difference of the arguments:

$$\varphi(x) = f(x) + \int_0^x K(x-t)\varphi(t)dt \qquad (\lambda = 1) \qquad (4.4.2.1)$$

Show that for equation (4.4.2.1) all iterated kernels and the resolvent kernel are also dependent solely on the difference x – t.

Let the functions f(x) and K(x) in (4.4.2.1) be original functions. Taking the Laplace transform of both sides of (4.4.2.1) and employing the product theorem (transform of a convolution), we get

$$\Phi(p) = F(p) + \tilde{K}(p)\Phi(p)$$

where

$$\varphi(x) = \Phi(p),$$
$$f(x) = F(p)$$
$$K(x) = \tilde{K}(p)$$

Whence

$$\Phi(p) = \frac{F(p)}{1-\tilde{K}(p)}, \qquad \tilde{K}(p) \neq 1 \qquad (4.4.2.2)$$

Taking advantage of the results of Problem 30, we can write the solution of the integral equation (4.4.2.1) in the form

$$\varphi(x) = f(x) + \int_0^x R(x-t)f(t)dt \qquad (4.4.2.3)$$

where R(x-t) is the resolvent kernel for the integral equation (4.4.2.1)

Taking the Laplace transform of both sides of equation (4.4.2.3), we obtain

$$\Phi(p) = F(p) + \tilde{R}(p)F(p)$$

where

$$R(x) = \tilde{R}(p)$$

Whence

$$\tilde{R}(p) = \frac{\Phi(p) - F(p)}{F(p)} \qquad (4.4.2.4)$$

Substituting into (4.4.2.4) the expression for F(p) from (4.4.2.2), we obtain

$$\tilde{R}(p) = \frac{\tilde{K}(p)}{1-\tilde{K}(p)} \qquad (4.4.2.5)$$

The original function of $\tilde{R}(p)$ will be the resolvent kernel of the integral equation (4.4.2.1)

Example: Find the resolvent kernel for a volterra integral equation with kernel K(x, t) = sin (x, t), λ = 1.

**Solution**: We have $\tilde{K}(p) = \frac{1}{p^2+1}$ by (4.4.2.5)

$$\tilde{R}(p) = \frac{\frac{1}{p^2+1}}{1-\frac{1}{p^2+1}} = \frac{1}{p^2} = x$$

Hence, the required resolvent kernel for the integral equation is

R(x, t; 1) = x – t

## Self Assessment Questions:

Find the resolvent kernels for Volterra – type integral equations with the kernels (λ = 1):

1. $K(x, t) = \sinh(x - t)$
2. $K(x, t) = e^{-(x-t)}$
3. $K(x, t) = e^{-(x-t)}\sin(x - t)$
4. $K(x, t) = \cosh(x - t)$
5. $K(x, t) = 2\cos(x - t)$

**Example**: With the aid of the resolvent kernel, find the solution of the integral equation

$$\varphi(x) = e^{x^2} + \int_0^x e^{x^2-t^2}\varphi(t)\,dt$$

Solution: The resolvent kernel of the kernel $K(x,t) = e^{x^2-t^2}$ for λ = 1 is $R(x,t;1) = e^{x-t}e^{x^2-t^2}$. By formula (4.4.1.13), the solution of the given integral equation is

$$\varphi(x) = e^{x^2} + \int_0^x e^{x-t}e^{x^2-t^2}e^{t^2}\varphi(t)\,dt = e^{x+x^2}$$

## 4.4.2.2. Self Assessment Questions:

Using the results of the preceding examples, find (by mans of resolvent kernels) solutions of eh following integral equations:

1. $\varphi(x) = e^x + \int_0^x e^{x-t}\varphi(t)dt$

2. $\varphi(x) = 1 - 2x - \int_0^x e^{x^2-t^2}\varphi(t)dt$

3. $\varphi(x) = \sin x + 2\int_0^x e^{x-t}\varphi(t)dt$

4. $\varphi(x) = x3^x - \int_0^x 3^{x-t}\varphi(t)dt$

5. $\varphi(x) = e^x \sin x + \int_0^x \frac{2+\cos x}{2+\cos t}\varphi(t)dt$

Note 1: The unique solvability of Volterra-type integral equations of the second kind

$$\varphi(x) = f(x) + \lambda\int_0^x K(x,t)\varphi(t)dt \qquad (4.4.2.6)$$

holds under considerably more general assumptions with respect to the function f(x) and the kernel K(x, t) than their continuity.

Theorem: The Volterra integral equation of the second kind (1), whose kernel K(x, t) and function f(x) belong, respectively, to spaces $L_2(\Omega_0)$ and $L_2(0, a)$, has one and only one solution in the space $L_2(0, a)$.
This solution is given by the formula

$$\varphi(x) = f(x) + \lambda\int_0^x R(x,t;\lambda)f(t)dt \qquad (4.4.2.7)$$

where the resolvent kernel R(x, t; λ) is determined by means of the series

$$R(x,t;\lambda) = \sum_{\nu=0}^{\infty}\lambda^{\nu}K_{\nu+1}(x,t) \qquad (4.4.2.8)$$

which is made up the iterated kernels and converges almost everywhere.
Note 2: In equation of uniqueness of solution of an integral equation, an essential role to played by the class of functions in

which the solution is sought (the class of summable, quadratically summable, continuous, etc., functions)

Thus, if the kernel K(x, t) of a Volterra equation is bounded when x varies in some finite interval (a, b) so that

$$| K(x, t)| \le M, M = const, x \notin (a, b)$$

and the constant term of f(x) is sum able in the interval (a, b),

then the Volterra equation has, for any value of λ, a unique summable solution φ (x) in the interval (a, b).

However, if we give up the requirement of summability of the solution, then the uniqueness theorem ceases to hold in the sense that the equation can have nonsummable solutions along with summable solutions.

P. S. Uryson ([29]) constructed elegant examples of integral equations (see Examples 1 and 2 below) which have summable and non summable solutions even when the kernel K(x, t) and the function f(x) are continuous.

For simplicity we consider f(x) ≡ 0 and examine the integral equation

$$\varphi(x) = \int_0^x K(x-t)\varphi(t)dt \tag{4.4.2.9}$$

where K(x, t) is a continuous function.

The only summable solution of equation (1) is φ (x) ≡ 0.

Example 1: Let $0 \le t \le x < a$ ($a > 0$, in particular $a = +\infty$).

$$K(x,t) = \frac{2}{\pi}\frac{xt^2}{x^6+t^2} \tag{4.4.2.10}$$

The function K(x, t) is even holomorphic everywhere, except at he point (0, 0). However, equation (4.4.2.9) with kernel (4.4.2.10) admits nonsummable solutions. Indeed, the equation

$$\psi(x) = \frac{2}{\pi}\int_0^x \frac{xt^2}{x^6+t^2}\psi(t)dt - \frac{2}{\pi}\frac{\arctan x^2}{x^2} \tag{4.4.2.11}$$

has a summable solution since the function

$$f(x) = -\frac{2}{\pi}\frac{\arctan x^2}{x^2}$$

is bounded and continuous everywhere except at the point x = 0.

The function

$$\varphi(x) = \begin{cases} 0, & x=0 \\ \psi(x) + \frac{1}{x^2} & x>0 \end{cases} \tag{4.4.2.12}$$

where ψ (x) is a solution of (4.4.2.11) will now be a nonsummable solution of (4.4.2.9) with kernel (4.4.2.10).

Indeed, for x > 0 we have

$$\int_0^x K(x,t)\varphi(t)dt = \frac{2}{\pi}\int_0^x \frac{xt^2}{x^6+t^2}\psi(t)dt + \frac{2}{\pi}\int_0^x \frac{xdt}{x^6+t^2} \qquad (4.4.3.13)$$

By virtue of equation (4.4.2.11), the first term on the right of (4.4.2.13) is

$$\psi(x) + \frac{2}{\pi}\frac{\arctan x^2}{x^2}$$

The second term yields

$$\frac{2}{\pi}\int_0^x \frac{xdt}{x^6+t^2} = \frac{2}{\pi}\left(\frac{1}{x^2}\arctan\frac{t}{x^3}\right)\Bigg|_{t=0}^{t=x} = \frac{2}{\pi x^2}\arctan\frac{1}{x^2} \qquad (x>0)$$

Thus

$$\int_0^x K(x,t)\varphi(t)dt = \psi(x) + \frac{2}{\pi}\frac{\arctan x^2}{x^2} + \frac{2}{\pi x^2}\arctan\frac{1}{x^2}$$

$$= \psi(x) + \frac{1}{x^2} = \varphi(x)$$

Which means hat the function φ (x) defined by (4.4.2.13) is a non sum able solution of equation (4.4.2.9) with kernel (4.4.2.11)

Example: 3 The equation

$$\varphi(x) = \int_0^x t^{x-t}\varphi(t)dt \qquad (0 \le x, t \le 1)$$

has a unique continuous solution φ (x) ≡ 0. By direct substitution we see that this equation also has an infinity of discontinuous solutions of the form

$$\phi(x) = Cx^{x-1}$$

where C is an arbitrary constant

## 4.4.3. The Method of Successive Approximations

Suppose we have a Volterra-type integral equation of the second kind:

$$\varphi(x) = f(x) + \lambda\int_0^x K(x-t)\varphi(t)dt \qquad \dots (4.4.3.1)$$

We assume that f(x) is continuous in [0, a] and the kernel K(x, t) is continuous for 0 ≤ x ≤ a, 0 ≤ t ≤ x.

Take some function $\varphi_0$ (x) continuous in [0, a]. Putting the function j $_0$(x) into the right side of (4.4.3.1) in place of $\varphi$ (x), we get

$$\varphi_1(x) = f(x) + \lambda \int_0^x K(x-t)\varphi_0(t)dt$$

The thus defined function $\varphi_1$(x) is also continuous in the interval [0, a]. Continuing the process, we obtain a sequence of functions

$\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x), \ldots$

where

$$\varphi_n(x) = f(x) + \lambda \int_0^x K(x-t)\varphi_{n-1}(t)dt$$

Under the assumptions with respect to f(x) and K (x, t), the sequence {$\varphi$, (x)} converges, as n $\to \infty$, to the solution $\varphi$(x) of he integral equation (4.4.3.1) (see [13]).

In particular, if for $\varphi_0$(x) we take f(x), then $\varphi_n$(x) will be the partial sums of the series (4.4.3.2), of Sec. 3, which defines the solution of the integral equation (4.4.3.1). A suitable choice of the "zero" approximation $\varphi_0$(x) can lead to a rapid convergence of the sequence { $\varphi_n$(x)} to the solution of the integral equation.

**Example**: Using the method of successive approximations, solve the integral equation

$$\varphi(x) = 1 + \int_0^x \varphi(t)\,dt$$

taking $\varphi_0(x) \equiv 0$.

Solution: Since $\varphi_0(x) \equiv 0$, it follows that $\varphi_1(x) = 1$. Then

$$\varphi_2(x) = 1 + \int_0^x 1\,dt = 1 + x$$

$$\varphi_3(x) = 1 + \int_0^x (1+t)\,dt = 1 + x + \frac{x^2}{2},$$

$$\varphi_4(x) = 1 + \int_0^x \left(1 + t + \frac{t^2}{2}\right)dt = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

Obviously

$$\varphi_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^{n-1}}{(n-1)!}$$

Thus, $\varphi_n(x)$ is the n[th] partial sum of the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}=e^x$. Whence it follows that $\varphi_n(x)\xrightarrow[n\to\infty]{}e^x$. It is easy to verify that the function $\varphi(x) = e^x$ is a solution of the given integral equation.

## 4.4.3.1. Self Assessment Questions:

Using the method of successive approximations, solve the following integral equations:

1. $\varphi(x)=x-\int_0^x (x-t)\varphi(t)\,dt, \qquad \varphi_0(x)\equiv 0.$

2. $\varphi(x)=1-\int_0^x (x-t)\varphi(t)\,dt, \qquad \varphi_0(x)\equiv 0.$

3. $\varphi(x)=1+\int_0^x (x-t)\varphi(t)\,dt, \qquad \varphi_0(x)=1.$

4. $\varphi(x)=x+1-\int_0^x \varphi(t)\,dt\,;(a)\varphi_0(x)=1, \quad (b)\varphi_0(x)=x+1$

5. $\varphi(x)=\frac{x^2}{2}+x-\int_0^x \varphi(t)\,dt;$

$(a)\ \ \varphi_0(x)=1, \qquad (b)\ \ \varphi_0(x)=x, \qquad (c)\ \ \varphi_0(x)=\frac{x^2}{2}+x$

Prove that the equation

$$\varphi(x)-\lambda\int_0^x K(x,t)\varphi(t)dt=0$$

has, for any λ, a unique solution $\varphi(x) \equiv 0$ in the class $L_2(0, a)$.
The method of successive approximations can also be applied to the solution of nonlinear Volterra integral equations of the form

$$y(x)=y_0+\int_0^x F[t,y(t)]dt \qquad (4.4.3.2)$$

or the more general equations

$$\varphi(x)=f(x)+\int_0^x F(x,t,\varphi(t))dt \qquad (4.4.3.3)$$

under extremely broad assumptions with respect to the functions F(x, t, z) and f(x). The problem of solving the differential equation

$$\frac{dy}{dx} = F(x, y), \qquad y|_{x=0} = y_0$$

reduces to an equation of the type (2). As in the case of linear integral equations, we shall seek the solution of equation (3) as the limit of the sequence $\{\varphi_n(x)\}$ where, for example, $\varphi_0(x) = f(x)$, and the following elements $\varphi_k(x)$ are computed successively from the formula

$$\varphi_k(x) = f(x) + \int_0^x F(x, t, \varphi_{k-1}(t))dt \qquad (k = 1, 2, \ldots) \qquad (4.4.3.4)$$

If f(x) and F(x, t, z) are quadratically summable and satisfy the conditions

$$|F(x, t, z_2) - F(x, t, z_1)| \le a(x, t)|z_2 - z_1| \qquad (4.4.3.5)$$

$$\left|\int_0^x F(x, t, f(t))dt\right| \le n(x) \qquad (4.4.3.6)$$

Where the functions a(x, t) and n(x) are such that in the main domain $(0 \le t \le x \le a)$

$$\int_0^a n^2(x)dx \le N^2, \qquad \int_0^a dx \int_0^x a^2(x, t)dt \le A^2 \qquad (4.4.3.7)$$

it follows that the nonlinear Volterra integral equation of the second kind (4.4.3.3) has a unique solution $\varphi(x) \in L_2(0, a)$ which is defined as the limit of $j_n(x)$ as $n \to \infty$:

$$\varphi(x) = \lim_{n \to \infty} \varphi_n(x)$$

where the functions $\varphi_n(x)$ are found from the recursion for mulas (4.4.3.4). For $\varphi_0(x)$ we can take any function in $L_2(0,a)$ (in particular, a continuous function), for which the condition (4.4.3.6) is fulfilled. Note that an apt choice of the zero approximation can facilitate solution of the integral equation.

**Example**: Using the method of successive approximations, solve the integral equation

$$\varphi(x) = \int_0^x \frac{1 + \varphi^2(t)}{1 + t^2} dt$$

taking as the zero approximation: (1) $\varphi_0(x) = 0$, (2) $\varphi_0(x) = x$.

$$\varphi_1(x) = \int_0^x \frac{dt}{1 + t^2} = \arctan x,$$

$$\varphi_2(x) = \int_0^x \frac{1 + \arctan^2 t}{1 + t^2} dt = \arctan x + \frac{1}{3}\arctan^3 x,$$

$$\varphi_3(x)=\int_0^x \frac{1+\left(\arctan t+\frac{1}{3}\arctan^3 t\right)^2}{1+t^2}\,dt$$

$$=\arctan x+\frac{1}{3}\arctan^3 x+\frac{2}{3\times 5}\arctan^5 x+\frac{1}{7\times 9}\arctan^7 x,$$

$$\varphi_4(x)=\int_0^x \frac{1+\varphi_3^2(t)}{1+t^2}\,dt=\arctan x+\frac{1}{3}\arctan^3 x+\frac{2}{3\times 5}\arctan^5 x+$$

$$\frac{17}{5\times 7\times 9}\arctan^7 x+\frac{38}{5\times 7\times 9^2}\arctan^9 x+\frac{134}{9\times 11\times 21\times 25}\arctan^{11} x+$$

$$\frac{4}{3\times 5\times 7\times 9\times 13}\arctan^{13} x+\frac{1}{7^2\times 9^2\times 15}\arctan^{15} x,\ldots$$

Denoting arctan x = u and comparing expressions for $\varphi_n(x)$ with the expansion

$$\tan u=\sum_{\nu=1}^{\infty}(-1)^{\nu-1}\frac{2^{2\nu}\left(2^{2\nu}-1\right)}{(2\nu)!}B_{2\nu}u^{2\nu-1},\quad |u|<\frac{\pi}{2}$$

where $B_\nu$ are Bernoulli numbers*, we observe that

$$\varphi_n(x)\xrightarrow[n\to\infty]{}\tan(\arctan x)=x$$

It can easily be verified that the function $\varphi(x)=x$ is a solution of the given integral equation.

## 4.4.4. Convolution-type equations

Let $\phi_1(x)$ and $\phi_2(x)$ be two continuous functions defined for $x\geq 0$. The convolution of these two functions is the function $\phi_3(x)$ defined by the equation

$$\varphi_3(x)=\int_0^x \varphi_1(x-t)\varphi_2(t)dt \tag{4.4.4.1}$$

This function, defined for $x\geq 0$, will also be a continuous function. If $\phi_1(x)$ and $\phi_2(x)$ are original functions for the Laplace transformation, then

$$L\phi_3=L\varphi_1.L\phi_2 \tag{4.4.4.2}$$

i.e., the transform of a convolution is equal to the product of the transforms of the functions (convolution theorem).

Let us consider the Volterra-type integral equation of the second kind

$$\varphi(x) = f(x) + \int_0^x K(x-t)\varphi(t)dt \qquad (4.4.4.3)$$

the kernel of which is dependent solely on the difference x-t. We shall call equation (4.4.4.3) an integral equation of the convolution type.

Let f(x) and K(x) be sufficiently smooth functions which, as $x \to \infty$, do not grow faster than the exponential function, so that

$$|f(x)| \le M_1 e^{s_1 x}, \qquad |K(x)| \le M_2 e^{s_2 x} \qquad (4.4.4.4)$$

Applying the method of successive approximations, we can show that in this case function $\phi(x)$ will also satisfy an upper bound of type (4.4.4.4)

$$|\varphi(x)| \le M_3 e^{s_3 x}$$

Consequently the Laplace transform of the functions f(x), K(x) and $\phi(x)$ can be found (it will be defined in the half plane Re p=s> max ($s_1$, $s_2$, $s_3$)).

Let

$$f(x) = F(p), \quad \varphi(x) = \Phi(p), \quad K(x) = \tilde{K}(p)$$

Taking the Laplace transform of both sides of (4.4.4.3) and employing the convolution theorem, we find

$$\Phi(p) = F(p) + \tilde{K}(p)\Phi(p) \qquad (4.4.4.5)$$

Whence

$$\Phi(p) = \frac{F(p)}{1-\tilde{K}(p)} \left(\tilde{K}(p) \ne 1\right)$$

The original function $\phi(x)$ for $\Phi(p)$ will be a solution of the integral equation (4.4.4.3).

Example: Solve the integral equation

$$\varphi(x) = \sin x + 2\int_0^x \cos(x-t)\varphi(t)dt$$

Solution: It is known that

$$\sin x = \frac{1}{p^2+1}, \qquad \cos x = \frac{p}{p^2+1}$$

Let $\phi(x) = \Phi(p)$. Taking the Laplace transform of both sides of the equation and taking account of the convolution theorem (transform of a convolution) we get

$$\Phi(p) = \frac{1}{p^2+1} + \frac{2p}{p^2+1}\Phi(p)$$

Whence

$$\Phi(p)\left[1 - \frac{2p}{p^2+1}\right] = \frac{1}{p^2+1}$$

or

$$\Phi(p) = \frac{1}{(p-1)^2} = xe^x$$

Hence, the solution of the given integral equation is

$\phi(x) = xe^x$

### 4.4.4.1. Self Assessment Questions:

Solve the following integral equations

1. $\varphi(x) = e^x - \int_0^x e^{x-t}\varphi(t)dt$

2. $\varphi(x) = x - \int_0^x e^{x-t}\varphi(t)dt$

3. $\varphi(x) = e^{2x} + \int_0^x e^{t-x}\varphi(t)dt$

4. $\varphi(x) = x - \int_0^x (x-t)\varphi(t)dt$

5. $\varphi(x) = \cos x - \int_0^x (x-t)\cos(x-t)\varphi(t)dt$

The Laplace transformation may be employed in the solution of systems of Volterra integral equations of the type

$$\varphi_i(x) = f_i(x) + \sum_{j=1}^{s}\int_0^x K_{ij}(x-t)\varphi_j(t)dt \qquad (i = 1, 2, \dots, s) \quad (4.4.4.6)$$

where $K_{ij}(x), f_i(x)$ are known continuous functions having Laplace transforms.

Taking the Laplace transform of both sides of (4.4.4.6), we get

$$\Phi_i(p) = F_i(p) + \sum_{i=1}^{s}\tilde{K}_{ij}(p)\Phi_j(p) \qquad (i = 1, 2, \dots, s) \qquad (4.4.4.7)$$

This is a system of linear algebraic equations in $\Phi_j(p)$. Solving it, we find $\Phi_j(p)$., the original functions of which will be the solution of the original system of integral equations (4.4.4.6).

**Example**: Solve the system of integral equations

$$\left.\begin{aligned}\varphi_1(x) &= 1-2\int_0^x e^{2(x-t)}\varphi_1(t)dt+\int_0^x \varphi_2(t)dt\\ \varphi_2(x) &= 4x-\int_0^x \varphi_1(t)dt+4\int_0^x (x-t)\varphi_2(t)dt\end{aligned}\right\} \qquad (4.4.8)$$

Solution: Taking transforms and using the theorem on the transform of a convolution, we get

$$\left.\begin{aligned}\Phi_1(p) &= \frac{1}{p}-\frac{2}{p-2}\Phi_1(p)+\frac{1}{p}\Phi_2(p)\\ \Phi_2(p) &= \frac{4}{p^2}-\frac{1}{p}\Phi_1(p)+\frac{4}{p^2}\Phi_2(p)\end{aligned}\right\}$$

Solving the system obtained for $\Phi_1(p)$ and $\Phi_2(p)$ we find

$$\Phi_1(p)=\frac{p}{(p+1)^2}=\frac{1}{p+1}-\frac{1}{(p+1)^2}$$

$$\Phi_2(p)=\frac{3p+2}{(p-2)(p+1)^2}=\frac{8}{9}\cdot\frac{1}{p-2}+\frac{1}{3}\cdot\frac{1}{(p+1)^2}-\frac{8}{9}\cdot\frac{1}{p+1}$$

The original functions for $\Phi_1(p)$ and $\Phi_2(p)$ are equal, respectively, to

$$\varphi_1(x)=e^{-x}-xe^{-x},$$

$$\varphi_2(x)=\frac{8}{9}e^{2x}+\frac{1}{3}xe^{-x}-\frac{8}{9}e^{-x}$$

The functions $\phi_1(x)$ and $\phi_2(x)$ are solutions of the original system of the integral equations (4.4.4.8)

## 4.4.4.2. Self Assessment Questions:

Solve the following systems of integral equations

1. $\varphi_1(x)=\sin x+\int_0^x \varphi_2(t)dt$,

$$\varphi_2(x)=1-\cos x-\int_0^x \varphi_1(t)dt$$

2. $\varphi_1(x) = e^{2x} + \int_0^x \varphi_2(t)dt$,

$$\varphi_2(x) = 1 - \int_0^x e^{2(x-t)}\varphi_1(t)dt$$

3. $\varphi_1(x) = e^x + \int_0^x \varphi_1(t)dt - \int_0^x e^{x-t}\varphi_2(t)dt$

$$\varphi_2(x) = -z - \int_0^x (x-t)\varphi_1(t)dt + \int_0^x \varphi_2(t)dt$$

4. $\varphi_1(x) = 1 - \int_0^x \varphi_2(t)dt$

$$\varphi_2(x) = \cos x - 1 + \int_0^x \varphi_3(t)dt$$

$$\varphi_3(x) = \cos x + \int_0^x \varphi_1(t)dt$$

## 4.4.5. Volterra Integral Equations with limits (x, +∞)

Integral equations of the form

$$\varphi(x) = f(x) + \int_x^\infty K(x-t)\varphi(t)dt \qquad (4.4.5.1)$$

which arise in a number of problems in physics can also be solved by means of the Laplace transformation. For this purpose, we establish the convolution theorem for the expressions

$$\int_0^\infty K(x-t)\varphi(t)dt \qquad (4.4.5.2)$$

It is known that for the Fourier transformation

$$F\left\{\int_{-\infty}^{+\infty} g(x-t)\psi(t)dt\right\} = \sqrt{2\pi}\, G(\lambda)\Psi(\lambda) \qquad (4.4.5.3)$$

where $G(\lambda), \Psi(\lambda)$ are Fourier transforms of the functions $g(x)$ and $\psi(x)$ respectively.
Put g(x)=K(x) i.e.,

$$g(x) = \begin{cases} 0, & x > 0 \\ K(x), & x < 0 \end{cases}$$

$$\psi(x) = \varphi_+(x) = \begin{cases} \varphi(x), & x > 0 \\ 0, & x < 0 \end{cases}$$

(4.4.5.4)

Then (4.4.5.3) can be rewritten as

$$\mathsf{F}\left\{\int_x^{+\infty} \mathrm{K}(x-t)\varphi(t)\mathrm{dt}\right\} = \sqrt{2\pi}\,\tilde{\mathrm{K}}_-(\lambda)_F\,\tilde{\Phi}_+(\lambda)_L \qquad (4.4.5.5)$$

To pass from the fourier transform to the Laplace transform, observe that

$$F_L(p) = \sqrt{2\pi}\,[F_+(ip)]_F \qquad (4.4.5.6)$$

Hence, from (4.4.5.5) and (4.4.5.6) we get

$$\mathsf{L}\left\{\int_x^{\infty} \mathrm{K}(x-t)\varphi(t)dt\right\} = \sqrt{2\pi}\,\left[\tilde{\mathrm{K}}_-(ip)\right]_F\,[\Phi_+(p)]_L \qquad (4.4.5.7)$$

We now express $\left[\sqrt{2\pi}\,\tilde{\mathrm{K}}_-(ip)\right]_F$ in terms of the Laplace transform:

$$\left[\sqrt{2\pi}\,\tilde{\mathrm{K}}_-(ip)\right]_F = \int_{-\infty}^{0} \mathrm{K}(x)e^{-px}dx = \int_0^{\infty} \mathrm{K}(-x)e^{px}dx$$

Putting K(-x)=K(x) we get

$$\left[\sqrt{2\pi}\,\tilde{\mathrm{K}}_-(ip)\right]_F = \tilde{\mathrm{K}}_L(-p) = \int_0^{\infty} \mathrm{K}(-x)e^{px}dx$$

And so

$$\mathsf{L}\left\{\int_x^{\infty} \mathrm{K}(x-t)\varphi(t)dt\right\} = \tilde{\mathrm{K}}_L(-p)\Phi_L(p) \qquad (4.4.5.8)$$

Let us now return to the integral equation (4.4.5.1). Taking the Laplace transform of both sides of (4.4.5.1), we obtain

$$\Phi(p) = F(p) + \tilde{\mathrm{K}}(-p)\Phi(p) \qquad (4.4.5.9)$$

or,

$$\Phi(p) = \frac{F(p)}{1-\tilde{\mathrm{K}}(-p)} \qquad \left(\tilde{\mathrm{K}}(-p) \neq 1\right) \qquad (4.4.5.10)$$

where

$$\breve{K}(-p)=\int_0^{\infty}K(-x)\,e^{px}\,dx \qquad (4.4.5.11)$$

The function

$$\varphi(x)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{F(p)}{1-\breve{K}(-p)}\,e^{px}\,dp \qquad (4.4.5.12)$$

is a particular solution of the integral equation (4.4.5.1). It must be stressed that the solution (4.4.5.9) or (4.4.5.12) is meaningful only iif the domains of analyticity of $\breve{K}(-p)$ and F(p) overlap.

**Example: Solve the integral equation**

$$\varphi(x)=x+\int_x^{\infty}e^{2(x-t)}\varphi(t)\,dt \qquad (4.4.5.13)$$

**Solution: In this case,** f(x)=x, K(x)=e$^{2x}$. Therefore

$$F(p)=\frac{1}{p^2},\quad \breve{K}(-p)=\int_x^{\infty}e^{-2t}e^{pt}\,dx=\frac{1}{2-p},\quad \mathrm{Re}\,p<2$$

Thus we obtain the following operator equation

$$\Phi(p)=\frac{1}{p^2}+\frac{1}{2-p}\Phi(p)$$

so that

$$\Phi(p)=\frac{p-2}{p^2(p-1)} \qquad (4.4.5.14)$$

Whence

$$\varphi(x)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{p-2}{p^2(p-1)}\,e^{px}\,dp \qquad (4.4.5.15)$$

Integral (4.4.5.15) may be evaluated from the Cauchy integral formula. The integrand function has a double pole p=0 and a simple pole p=1. which appears for $\gamma>1$; this is connected with including or not including in the solution of equation (4.4.5.13) the solution of the corresponding homogeneous equation

$$\varphi(x) = \int_x^\infty e^{2(x-t)}\varphi(t)dt$$

Let us find the residues of the integrand function at its poles:

$$\underset{p=0}{res}\left(\frac{p-2}{p^2(p-1)}e^{px}\right) = 2x+1, \qquad \underset{p=1}{res}\left(\frac{p-2}{p^2(p-1)}e^{px}\right) = -e^x$$

Consequently the solution of the integral equation (4.4.5.13) is $\varphi(x) = 2x + 1 + ce^x$ (C is an arbitrary constant)

## 4.4.5.1.Self Assessment Questions:

Solve the integral equations

1. $\varphi(x) = e^{-x} + \int_x^\infty \varphi(t)dt$
2. $\varphi(x) = e^{-x} + \int_x^\infty e^{x-t}\varphi(t)dt$
3. $\varphi(x) = \cos x + \int_x^\infty e^{x-t}\varphi(t)dt$
4. $\varphi(x) = 1 + \int_x^\infty e^{\alpha(x-t)}\varphi(t)dt \qquad (\alpha > 0)$

## 4.4.6. Volterra Integral Equations of the First Kind

Suppose we have a Volterra integral equation of the first kind

$$\int_0^x K(x,t)\varphi(t)dt = f(x), \qquad f(0) = 0 \qquad (4.4.6.1)$$

where $\phi(x)$ is the unknown function.

Suppos that $K(x,t)$, $\frac{\partial K(x,t)}{\partial x}$, $f(x)$ and $f'(x)$ are continuous for $0 \le x \le a$, $0 \le t \le x$. Differentiating both sides of (4.4.6.1) with respect to x, we obtain

$$K(x,x)\varphi(x) + \int_0^x \frac{\partial K(x,t)}{\partial x}\varphi(t)dt = f'(x) \qquad (4.4.6.2)$$

Any continuous solution $\phi(x)$ of equation (4.4.6.1) for $0 \le x \le a$ obviously satisfies equation (4.4.6.2) as well. Conversely, any continuous solution of equation (4.4.6.2) for $0 \le x \le a$ satisfies equation (4.4.6.1) too.

If K(x,x) does not vanish at any point of the basic interval [0,a] then equation (4.4.6.2) can be rewritten as

$$\varphi(x) = \frac{f'(x)}{K(x,x)} - \int_0^x \frac{K'_x(x,t)}{K(x,x)}\varphi(t)dt \qquad (4.4.6.3)$$

which means it reduces to a Volterra type integral equation of the second kind which has already been considered.

If K(x,x)=0 then it is sometimes useful to differentiate (4.4.6.2) onece again w.r.t x and so on.

Example: Solve he integral equation

$$\int_0^x \cos(x-t)\varphi(t)dt = x \qquad (4.4.6.4)$$

Solution: The function f(x)=x. K(x,t)=cos(x-t) satisfy the above formulated conditions of continuity and differentiability.
Differentiating both sides of (4.4.6.4) w.r.t x we get

$$\varphi(x)\cos 0 - \int_0^x \sin(x-t)\varphi(t)dt = 1$$

or

$$\varphi(x) = 1 + \int_0^x \sin(x-t)\varphi(t)dt \qquad (4.4.6.5)$$

Equation (4.4.6.5) is an integral equation of the second kind of the convolution type.
We find its solution by applying the Laplace transformation

$$\Phi(p) = \frac{1}{p} + \frac{1}{p^2+1}\Phi(p)$$

$$\Phi(p) = \frac{p^2+1}{p^3} = \frac{1}{p} + \frac{1}{p^3} = 1 + \frac{x^2}{2}$$

The function $\varphi(x) = 1 + \frac{x^2}{2}$ will be a solution of equation (4.4.6.5) and hence of the original equation (4.4.6.4) as well. This is readily seen by direct verification.

## 4.4.6.1. Self Assessment Questions:

Solve the following integral equations of the first kind by first reducing them to integral equations of the second kind.

1. $\int_0^x e^{x-t}\varphi(t)dt = \sin x$

2. $\int_0^x 3^{x-t}\varphi(t)dt = x$

3. $\int_0^x a^{x-t}\varphi(t)dt = f(x), \quad f(0) = 0$

4. $\int_0^x (1 - x^2 + t^2)\varphi(t)dt = \frac{x^2}{2}$

5. $\int_0^x (2 + x^2 - t^2)\varphi(t)dt = x^2$

## 4.4.7. Euler Integrals

The gama function or Euler's integral of the second kind is the function $\Gamma(x)$ defined by the equality

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \tag{4.4.7.1}$$

where x is any complex number, Re(x)>0. For x=1 we get

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1 \tag{4.4.7.2}$$

Integrating by parts, we obtain from (4.4.7.1)

$$\Gamma(x) = \frac{1}{x}\int_0^\infty e^{-t} t^x dt = \frac{\Gamma(x+1)}{x} \tag{4.4.7.3}$$

This equation expresses the basic property of a gama function:

$$\Gamma(x+1) = x\Gamma(x) \tag{4.4.7.4}$$

Using (4.4.7.2), we get

$$\Gamma(2) = \Gamma(1+1) = 1.\Gamma(1) = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2.\Gamma(2) = 2!$$

$$\Gamma(4) = \Gamma(3+1) = 3.\Gamma(3) = 3!$$

and, generally, for positive integral n

$$\Gamma(n) = (n-1)! \qquad (4.4.7.5)$$

We know that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

Putting $x = t^{\frac{1}{2}}$ here, we obtain

$$\int_0^\infty e^{-t} t^{\frac{1}{2}-1}\,dt = \sqrt{\pi}$$

Taking into account expression (4.4.7.1) for the gamma function, we can write this equation as

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Whence, by means of the basic property of a gamma function by (4.4.7.4), we find $\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$ ,

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1\times 3}{2^2}\sqrt{\pi} \text{ and so on.}$$

Generally, it will readily be seen that the following equality holds:

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1\times 3\times 5\ldots(2n-1)}{2^n}\sqrt{\pi} \qquad (4.4.7.6)$$

Knowing the value of the gamma function for some value of the argument, we can compute from (4.4.7.3) the valuye of the function for an argument diminished by unity. Fro example

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

For this reason

$$\Gamma\left(\frac{1}{2}\right)=\frac{\Gamma\left(\frac{3}{2}\right)}{\frac{1}{2}}=\sqrt{\pi} \qquad (4.4.7.7)$$

Acting in similar fashion, we find

$$\Gamma\left(-\frac{1}{2}\right)=\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}=-2\sqrt{\pi},$$

$$\Gamma\left(-\frac{3}{2}\right)=\frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}}=\frac{4}{3}\sqrt{\pi},$$

$$\Gamma\left(-\frac{5}{2}\right)=-\frac{8}{15}\sqrt{\pi} \text{ and so on.}$$

It is easy to verify that $\Gamma(0)=\Gamma(-1)=...=\Gamma(-n)=...=\alpha$ Above we defined $\Gamma(x)$ for Re x>0. The indicted method for computing $\Gamma(x)$ extends this function into the left half plane, where $\Gamma(x)$ is defined everywhere excepot at the points x=-n (n is positive integer and 0).

Note also the following relations

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin \pi x}, \qquad (4.4.7.8)$$

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right)=2^{1-2x}\pi^{1/2}\Gamma(2x) \qquad (4.4.7.9)$$

and generally

$$\Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\Gamma\left(x+\frac{2}{n}\right)...\Gamma\left(x+\frac{n-1}{n}\right)=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx}\Gamma(nx)$$

(Gauss-Legendre multiplication theorem)

The gamma function was represented by Wereirstrass by means of the equation

$$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\} \qquad (4.4.7.10)$$

where

$$\gamma = \lim_{m\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{m}-\ln m\right)=0.57721\ldots$$

is Euler's constant. From (4.4.7.10) it is evident that the function $\Gamma(z)$ is analytic everywhere except at z=0, z=-1, z=-2... where it

has simple poles.
The following is Euler's formula which is obtained from (4.4.7.10)

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right\} \qquad (4.4.7.11)$$

It holds everywhere except at z=0, z=-1, z=-2...

We introduce Euler's integral of the first kind $B(p,q)$, the so called beta function:

$$B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx \quad (\operatorname{Re} p>0,\quad \operatorname{Re} q>0) \qquad (4.4.7.12)$$

The following equality holds (it establishes a relationship between the Euler integrals of the first and second kinds)

$$B(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \qquad (4.4.7.13)$$

## 4.4.7.1. Self Assessment Questions:

1. Show that $\Gamma'(1)=-\gamma$

2. Show that for Re z>0

$$\Gamma(z)=\int_0^1\left(\ln\frac{1}{x}\right)^{z-1}dx$$

3. Show that

$$\frac{\Gamma'(1)}{\Gamma(1)}-\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}=2.\ln 2$$

4. Prove that

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\times 2\ldots(n-1)}{z(z+1)\ldots(z+n-1)} n^z$$

## 4.5. Abel's Problem
## Abel's Integral Equation and Its Generalizations

A particle is constrained to move under the force of gravity in a vertical plane $(\xi, \eta)$ along a certain path. It is required to determine this path so that the particle having started from rest at a point on the curve with ordinate x, reaches the $\xi$ axis in time $t = f_1(x)$ where $f_1(x)$ is a given function.

The absolute velocity of a moving particle is $v = \sqrt{2g(x-\eta)}$. Denote by $\beta$ the angle of inclination of the tangent to the $\xi$ axis. Then we will have

$$\frac{d\eta}{dt} = -\sqrt{2g(x-\eta)}\sin\beta$$

whence

$$dt = -\frac{d\eta}{\sqrt{2g(x-\eta)}\sin\beta}$$

Integrating from 0 to x and denoting $\frac{1}{\sin\beta} = \varphi(\eta)$, we get Abel's equation

$$\int_0^x \frac{\varphi(\eta)d\eta}{\sqrt{x-\eta}} = -\sqrt{2g}\, f_1(x)$$

Denoting $-\sqrt{2g}\, f_1(x)$ by f(x), we finally obtain

$$\int_0^x \frac{\varphi(\eta)}{\sqrt{x-\eta}} d\eta = f(x) \qquad (4.5.1)$$

where $\phi(x)$ is the required function and f(x) is the given function. After finding $\phi(\eta)$ we can form the equation of the curve. Indeed

$$\varphi(\eta) = \frac{1}{\sin\beta}$$

whence $\eta = \Phi(\beta)$

Further $d\xi = \dfrac{d\eta}{\tan\beta} = \dfrac{\Phi'(\beta)d\beta}{\tan\beta}$

Whence

$$\xi = \int \frac{\Phi'(\beta)d\beta}{\tan\beta} = \Phi_1(\beta)$$

and consequently the required curve is defined by the parametric equations

$$\left.\begin{aligned} \xi &= \Phi_1(\beta) \\ \eta &= \Phi(\beta) \end{aligned}\right\} \qquad (4.5.2)$$

Thus Abel's problem reduces to a solution of the integral equation

$$f(x) = \int_0^x \mathrm{K}(x,t)\varphi(t)dt$$

with given kernel $K(x,t)$ given function $f(x)$ and unknown function $\phi(x)$; in other words, it reduces to finding a solution of the Volterra integral equation of the first kind.

The following somewhat more general equation is also called Abel's equation

$$\int_0^x \frac{\varphi(t)}{(x-t)^\alpha} dt = f(x) \qquad (4.6.3)$$

where $\alpha$ is a constant, $0 < \alpha < 1$ (Abel's generalized equation). We will consider that the function f(x) has a continuous derivative on some interval [0,a]. Note that for $\alpha \geq \dfrac{1}{2}$ the kernel of equation (4.5.3) is quadratic ally non integrable, i.e., it is not an L2 function. However, equation (4.5.3) has a solution which may be found in the following manner.

Suppose equation (4.5.3) has a solution. Replace x by s in the equation and multiply both sides of the resulting equality by $\dfrac{ds}{(x-s)^{1-\alpha}}$ and integrate with respect to s from 0 to x:

$$\int_0^x \frac{dx}{(x-s)^{1-\alpha}} \int_0^s \frac{\varphi(t)}{(s-t)^\alpha} dt = \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}} ds \qquad (4.5.4)$$

Changing the order of integration on the left, we obtain

$$\int_0^x \varphi(t)dt \int_t^x \frac{ds}{(x-s)^{1-\alpha}(s-t)^{\alpha}} = F(x) \tag{4.5.5}$$

where

$$F(x) = \int_0^x \frac{f(s)}{(x-s)^{1-\alpha}} ds$$

(4.5.6)

In the inner integral make the substitution s = t + y (x-t):

$$\int_t^x \frac{ds}{(x-s)^{1-\alpha}(s-t)^{\alpha}} = \int_0^1 \frac{dy}{y^{\alpha}(1-y)^{1-\alpha}} = \frac{\pi}{\sin\alpha\,\pi}$$

Then from equation (4.5.5) we have

$$\int_0^x \varphi(t)dt = \frac{\sin\alpha\,\pi}{\pi} F(x)$$

or

$$\varphi(x) = \frac{\sin\alpha\,\pi}{\pi} F'(x) = \frac{\sin\alpha\,\pi}{\pi}\left(\int_0^x \frac{f(s)}{(x-s)^{1-\alpha}} ds\right)_x \tag{4.5.7}$$

Thus the only solution of equation (4.5.3) is given by formula (4.5.7) which via integration bny parts can also be re written in the form

$$\varphi(x) = \frac{\sin\alpha\,\pi}{\pi}\left[\frac{f(0)}{x^{1-\alpha}} + \int_0^x \frac{f'(s)}{(x-s)^{1-\alpha}} ds\right] \tag{4.5.8}$$

This solution has physical meaning only when its absolute value is not less that 1 (since $\varphi(x) = \dfrac{1}{\sin\beta}$)

We will show that in the case f(x)=C=const, the solution of Abel's problem is a cycloid. (The tautochrone problem: to find the curve along which a particle moving under gravity without friction reaches its lowest position in the same time, irrespective of is initial position)

In this case $\alpha = \dfrac{1}{2}$. Hence, by formula (4.5.8)

$$\varphi(x) = \frac{1}{\pi}\frac{C}{\sqrt{x}}$$

And therefore

$$\sin\beta = \frac{\pi\sqrt{\eta}}{C}$$

Whence $\eta = \frac{C^2}{\pi^2}\sin^2\beta$

Further

$$d\xi = \frac{d\eta}{\tan\beta} = \frac{C^2}{\pi^2}\frac{2\sin\beta\cos\beta}{\tan\beta}d\beta$$

$$= \frac{C^2}{\pi^2}(1+\cos 2\beta)d\beta$$

$$\xi = \frac{C^2}{\pi^2}\left(\beta + \frac{1}{2}\sin 2\beta\right) + C_1$$

Finally,

$$\xi = \frac{C^2}{\pi^2}\left(\beta + \frac{1}{2}\sin 2\beta\right) + C_1$$

$$\eta = \frac{C^2}{2\pi^2}(1-\cos 2\beta)$$

(parametric equations of the cycloid)

## 4.5.1. Self Assessment Questions:

1. Show that when $f(x) = C\sqrt{x}$ the solution of Abel's problem will be straight lines.

Solve the following integral equations

2. $\int_0^x \frac{\varphi(t)dt}{(x-t)^\alpha} = x^n \qquad (0 < \alpha < 1)$

3. $\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}} = \sin x$

4. $\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}} = e^x$

5. $\int_0^x \frac{\varphi(t)dt}{\sqrt{x-t}} = x^{\frac{1}{2}}$

6. $\iint_D \frac{\varphi(x,y)dxdy}{\sqrt{(y_0-y)^2-(x_0-x)^2}} = f(x_0-y_0)$

## 4.6. Volterra integral equations of the first kind of the Convolution type

An integral equation of the first kind

$$\int_0^x K(x-t)\varphi(t)dt = f(x) \qquad (4.6.1)$$

whose kernel K(x,t) is dependent solely on the difference x-t of arguments will be called an integral equation of the first kind of convolution type.

This class of equations includes, for instance, the generalized Abel equation.

Let us consider a problem that leads to a Volterra integral equation of the convolution type.

A shop buys and sells a variety of commodities. It is assumed that:

(1) buying and selling are continuous processes and purchased goods are put on sale at once.

(2) the shop[ acquires each new lot of any type of goods in quantities which it can sell in a time interval T, the same for all purchases;

(3) each new lot of goods is sold uniformly over time T.

The shop initiates the sale of a new batch of goods, the total cost of which is unity. It is required to find the law $\phi(t)$ by which it should make purchases so that the cost of goods on hand should be constant.

Solution: Let the cost of the original goods on hand at time t be equal to K(t) where

$$K(t) = \begin{cases} 1-\frac{t}{T}, & t \le T \\ 0, & t > T \end{cases}$$

Let us suppose that in the time interval between $\tau$ and $\tau + d\tau$ goods are bought amounting to the sum of $\phi(\tau)d\tau$. This reserve diminishes (due to sales) in such a manner that the cost of the remaining goods at time $t > \tau$ is equal to $K(t-\tau)\phi(\tau)d\tau$

Thus, $\phi(t)$ should satisfy the integral equation

$$1-K(t)=\int_0^t K(t-\tau)\phi(\tau)d\tau$$

We have thus obtained a Volterra integral equation of the first kind of the convolution type.
Let f(x) and K(x) be original functions and let

$$f(x)=F(p),\ K(x)=\tilde{\mathrm{K}}(p)\ \phi(x)=\phi(p)$$

Taking the Laplace transform of both sides of equation (4.6.1) and utilizing the convolution theorem, we will have

$$\overline{K}(p)\phi(p)=F(p) \qquad (4.6.2)$$

hence $$\Phi(p)=\frac{F(p)}{\tilde{\mathrm{K}}(p)} \qquad \left(\tilde{\mathrm{K}}(p)\neq 0\right) \qquad (4.6.3)$$

The original function $\phi(x)$ for the function $\phi(p)$ defined by (4.6.3) will be a solution of the integral equation (4.6.1).

Example: Solve the integral equation

$$\int_0^x e^{x-t}\varphi(t)dt=x \qquad (4.6.4)$$

Solution: Taking the Lapalce transform on both sides of (4.6.4), we obtain

$$\frac{1}{p-1}\Phi(p)=\frac{1}{p^2} \qquad (4.6.5)$$

hence $\Phi(p)=\frac{p-1}{p^2}=\frac{1}{p}-\frac{1}{p^2}=1-\mathrm{x}$

The function $\phi(x)=1-x$ is a solution of equation (4.6.4)

## 4.6.1. Self Assessment Questions:

Solve the integral equations

1. $\int_0^x \cos(x-t)\varphi(t)dt=\sin x$

2. $\int_0^x e^{x-t}\varphi(t)dt=\sinh x$

3. $\int_0^x (x-t)^{\frac{1}{3}} \varphi(t)dt = x^{\frac{5}{2}}$

4. $\int_0^x e^{2(x-t)} \varphi(t)dt = \sin x$

5. $\int_0^x e^{x-t} \varphi(t)dt = x^2$

Note: If $K(x,x) = K(0) \neq 0$ , then equation (4.6.1) definitely has a solution. If the kernel $K(x,t)$ becomes identically zero for t=x, yet the equation has a solution.
As has already been pointed out before, a necessary condition for the existence of a continuous solution of an integral equation of the form

$$\int_0^x \frac{(x-t)^{n-1}}{(n-1)!} \varphi(t)dt = f(x) \qquad \ldots (4.6.6)$$

consists in the function f(x) having continuous derivatives up to the nth order inclusive and in all its n-1 first derivatives vanishing for x=0.

This model equation (4.6.6) points to the necessity of matching the orders of vanishing of the kernel for t=x and of the right side f(x) for x=0 (the right side must exceed the left side by at least unity).

Consider the integral equation

$$\int_0^x (x-t)\varphi(t)dt = x \qquad (4.6.7)$$

Here f(x)=x, n=2. Obviously, f(x) ha derivatives of all orders, but its first derivative $f'(x) = 1 \neq 0$; that is the necessary condition is not fulfilled.
Taking the Laplace transform of both sides of (4.6.7) in formal fashion, we get

$$\frac{1}{p^2}\Phi(p) = \frac{1}{p^2}$$

whence $\Phi(p) = 1$
This is the transform of the $\delta$ function $\delta(x)$
Recall that $\delta(x) = 1$

$$\delta^{(m)}(x) = p^m$$

where m is an integer>=0.
Thus, the solution of the integral equation (4.6.7) is the $\delta$ function.
$\phi(x) = \delta(x)$
This is made clear by direct verification if we take into account that

the convolution of the $\delta$ function and any other smooth function g(x) is defined as

$$g(x) * \delta(x) = g(x)$$

$\delta^{(k)}(x) * g(x) = g^{(k)}(x)$ (k=1,2,3,….)

Indeed, in our case $g(x) = K(x) = x$ and

$$\int_0^x K(x-t)\delta(t)dt = K(x) = x$$

Thus the solution of equation (4.6.7) exists, but now in the class of generalized functions.

## 4.6.2. Self Assessment Questions:

1. $\int_0^x (x-t)\varphi(t)dt = x^2 + x - 1$

2. $\int_0^x (x-t)\varphi(t)dt = \sin x$

3. $\int_0^x (x-t)^2 \varphi(t)dt = x^2 + x^3$

4. $\int_0^x \sin(x-t)\varphi(t)dt = x + 1$

5. $\int_0^x \sin(x-t)\varphi(t)dt = 1 - \cos x$

Integral equations of the first kind with logarithmic kernel

$$\int_0^x \varphi(t)\ln(x-t)\,dt = f(x), \qquad f(0)=0 \tag{4.6.8}$$

can also be solved by means of the Laplace transformation.
We know that

$$x^{\nu} = \frac{\Gamma(\nu+1)}{p^{\nu+1}} \qquad (\operatorname{Re}\nu > -1) \tag{4.6.9}$$

Differentiate the relation (4.6.9) with respect to $\nu$

$$x^{\nu}\ln x = \frac{1}{p^{\nu+1}}\frac{d\Gamma(\nu+1)}{d\nu} + \frac{1}{p^{\nu+1}}\ln\frac{1}{p}\Gamma(\nu+1)$$

or $$x^{\nu}\ln x = \frac{\Gamma(\nu+1)}{p^{\nu+1}}\left[\frac{\frac{d\Gamma(\nu+1)}{d\nu}}{\Gamma(\nu+1)} + \ln\frac{1}{p}\right] \tag{4.610}$$

For $\nu = 0$ we have

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma$$

where $\gamma$ is Euler's constant and formula (4.6.10) takes the form

$$\ln x = \frac{1}{p}(-\gamma - \ln p) = -\frac{\ln p + \gamma}{p} \tag{4.6.11}$$

Let $\varphi(x) = \Phi(p)$, $f(x) = F(p)$. Taking the Laplace transform of both sides of (4.6.8) and utilizing formula (4.6.11), we get

$$-\Phi(p)\frac{\ln p + \gamma}{p} = F(p)$$

whence $$\Phi(p) = -\frac{pF(p)}{\ln p + \gamma} \tag{4.6.12}$$

Let us write $\Phi(p)$in the form

$$\Phi(p) = -\frac{p^2F(p) - f'(0)}{p(\ln p + \gamma)} - \frac{f'(0)}{p(\ln p + \gamma)} \tag{4.6.13}$$

Since f(0)=0, it follows that

$$\rho^2 F(p) - f'(0) = f''(x) \tag{4.6.14}$$

Let us return to formula (4.6.9) and write it in the form

$$\frac{x^{\nu}}{\Gamma(\nu+1)} = \frac{1}{p^{\nu+1}} \tag{4.6.9'}$$

Integrate both sides of (4.6.9′) w.r.t to $\nu$ from 0 to $\alpha$. This yields

$$\int_0^\infty \frac{x^\nu}{\Gamma(\nu+1)}d\nu = \int_0^\infty \frac{d\nu}{p^{\nu+1}} = \frac{1}{p\ln p}$$

By the similarity theorem

$$\int_0^\infty \frac{x^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}d\nu = \frac{1}{p\ln(ap)} = \frac{1}{p(\ln p+\ln a)}$$

If we put $a=e^\gamma$ then

$$\int_0^\infty \frac{x^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}d\nu = \frac{1}{p(\ln p+\gamma)} \qquad (4.6.15)$$

Take advantage of equality of (4.6.13). By virtue of (4.6.15)

$$\frac{f'(0)}{p(\ln p+\gamma)} = f'(0)\int_0^\infty \frac{x^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}d\nu$$

Taking into account (4.6.14) and (4.6.15), the first term on the right of (4.6.13) may be regarded as a product of transforms. To find its original function, take advantage of the convolution theorem.

$$\frac{p^2F(p)f'(0)}{p(\ln p+\gamma)} = \int_0^x f''(t)\left(\int_0^\infty \frac{(x-t)^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}d\nu\right)dt$$

Thus, the solution $\phi(x)$ of the integral equation (4.6.8) will have the form

$$\varphi(x) = -\int_0^x f''(t)\left(\int_0^\infty \frac{(x-t)^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}d\nu\right)dt - f'(0)\int_0^\infty \frac{x^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}d\nu$$

where $\gamma$ is Euler's constant.

In particular, for f(x)=x we get

$$\varphi(x) = -\int_0^\infty \frac{x^\nu e^{-\gamma\nu}}{\Gamma(\nu+1)}d\nu$$

The convolution theorem can also be used for solving non linear Volterra integral equations of the type

$$\varphi(x) = f(x) + \lambda\int_0^x \varphi(t)\varphi(x-t)dt \qquad (4.6.16)$$

Let $\phi(x) = \Phi(p),\ f(x) = F(p)$

Then by virtue of equation (4.6.16)

$$\Phi(p) = F(p) + \lambda\Phi^2(p)$$

whence

$$\Phi(p)=\frac{1\pm\sqrt{1-4\lambda\, F(p)}}{2\lambda}$$

The original function of , if it exists, will be a solution of the integral equation (4.6.16).

Example: Solve the integral equation

$$\int_0^x \varphi(t)\varphi(x-t)dt=\frac{x^3}{6} \qquad (4.6.17)$$

Solution: Let $\phi(x)=\Phi(p)$ . Taking the Laplace transform of both sides of (4.6.17), we get

$$\Phi^2(p)=\frac{1}{p^4}$$

whence $\Phi(p)=\pm\frac{1}{p^2}$

The function $\phi_1(x)=x, \phi_2(x)=-x$ will be solution of the equation (4.6.17).

## 4.6.3. Self Assessment Questions:

Solve the integral equations

1. $2\varphi(x)-\int_0^x \varphi(t)\varphi(x-t)dt=\sin x$

2. $\varphi(x)=\frac{1}{2}\int_0^x \varphi(t)\varphi(x-t)dt-\frac{1}{2}\sinh x$

## 4.7. Let us sum up

In this block we have discussed the following points

1. We have given the occurrence of integral equation and its basic concept
2. We have also shown the relationship between a differential equation and an integral equation.
3. We have also defined Volterra integral equation, its resolvent kernel and solution of Volterra equation by resolvent kernel.
4. We have given the solution of Volterra equation by successive approximation.
5. The concept of Euler's integral, Abel's Problem and Abel's integral equation are also given.
6. Finally we have discussed the Volterra equation of the first kind of the convolution type

# BLOCK 5
# FREDHOLM INTEGRAL EQUATIONS

**Structure**

## 5.0 Introduction

In the theory of integral equation the well known theorems of linear algebra which are related to solution of system of linear algebraic equations play an important role.

### 5.1.1 Objectives

After reading this block, you will be able to

- understand the concept of Fredholm integral equation
- Understand the method of Fredholm determinant, iterated kernel
- Construct resolvent kernel using iterated kernels
- Know the concept of degenerate kernel, Characteristic number and eigen functions

- Know the solution of homogeneous integral equation and non-homogeneous symmetric system.

5.1.2.Fredholm integral equation

## 5.1.2.1. Fredholm Equations of the Second Kind. Fundamentals

A linear Fredholm integral equation of the second kind is a equation of the form

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = f(x) \qquad (5.1.2.1.1)$$

where $\varphi(x)$ is the unknown function, $K(x,t)$ and $f(x)$ are known functions, x and t are real variables varying in the interval (a,b) and $\lambda$ is a numerical factor.

The function $K(x,t)$ is called the kernel of the integral equation (5.1.2.1.1); it is assumed that the kernel $K(x,t)$ is defined in the square $\Omega\{a \le x \le b, a \le t \le b\}$ in the $(x,t)$ plane and is continuous in $\Omega$ or its discontinuities are such that the double integral

$$\int_a^b \int_a^b |K(x,t)|^2 dx\, dt$$

has a finite value.

If $f(x) \neq 0$ equation (2.1) is non homogoneous; but if $f(x) \equiv 0$, then (5.1.2.1.1) takes the form

$$\varphi(x) - \lambda \int_a^b K(x,t)\varphi(t)dt = 0 \qquad (5.1.2..1.2)$$

and is called homogoneous.

The limits of integration, a and b in equations (2.1.1) and (2.1.2) is any function $\varphi(x)$ which, when substituted into the equations, reduces them to identities in $x \in (a,b)$

**Example**: Show that the function $\varphi(x) = \sin\frac{\pi x}{2}$ is a solution of the Fredholm type integral equation

$$\varphi(x) - \frac{\pi^2}{4}\int_0^1 K(x,t)\varphi(t)dt = \frac{x}{2}$$

where the kernel is of the form

$$K(x,t)=\begin{cases}\dfrac{x(2-t)}{2}, & 0\le x\le t\\[2mm] \dfrac{t(2-x)}{2}, & t\le x\le 1\end{cases}$$

Solution: Write the left hand side for the equation as

$$\varphi(x)-\frac{\pi^2}{4}\int_0^1 \mathrm{K}(x,t)\varphi(t)dt=$$

$$=\varphi(x)-\frac{\pi^2}{4}\left\{\int_0^x \mathrm{K}(x,t)\varphi(t)dt+\int_x^1 \mathrm{K}(x,t)\varphi(t)dt\right\}$$

$$=\varphi(x)-\frac{\pi^2}{4}\left\{\int_0^x \frac{t(2-x)}{2}\varphi(t)dt+\int_x^1 \frac{x(2-t)}{2}\varphi(t)dt\right\}$$

$$=\varphi(x)-\frac{\pi^2}{4}\left\{\frac{2-x}{2}\int_0^x t\varphi(t)dt+\frac{x}{2}\int_x^1(2-t)\varphi(t)dt\right\}$$

Substituting the function $\sin\frac{\pi x}{2}$ in place of $\varphi(x)$ into this expression, we get

$$\sin\frac{\pi x}{2}-\frac{\pi^2}{4}\left\{(2-x)\int_0^x t\frac{\sin\frac{\pi t}{2}}{2}dt+x\int_x^1(2-t)\frac{\sin\frac{\pi t}{2}}{2}dt\right\}$$

$$=\sin\frac{\pi x}{2}-\frac{\pi^2}{4}\left\{(2-x)\left(-\frac{t}{\pi}\cos\frac{\pi t}{2}+\frac{2}{\pi^2}\sin\frac{\pi t}{2}\right)\Bigg|_{t=0}^{t=x}+\right.$$

$$\left.+x\left[-\frac{2-t}{2}\cos\frac{\pi t}{2}-\frac{2}{\pi^2}\sin\frac{\pi t}{2}\right]_{t=x}^{t=1}\right\}=\frac{x}{2}$$

Thus, we have $\frac{x}{2}=\frac{x}{2}$, which, by definition, implies that $\varphi(x)=\sin\frac{\pi x}{2}$ is a solution of the given integral equation.

Check to see which of the given functions are solutions of the indicated integral equations.

## 5.1.2.1.1.Self Assessment Questions

1. $\varphi(x)=1,\quad \varphi(x)+\int_0^1 x\left(e^{xt}-1\right)\varphi(t)dt=e^x-x$

2. $\varphi(x)=e^x\left(2x-\frac{2}{3}\right),\quad \varphi(x)+2\int_0^1 e^{x-t}\varphi(t)dt=2xe^x$

3. $\varphi(x)=1-\frac{2\sin x}{1-\frac{\pi}{2}},\quad \varphi(x)-\int_0^{\pi}\cos(x+t)\varphi(t)dt=1$

4. $\varphi(x)=e^x,\quad \varphi(x)+\lambda\int_0^1 \sin xt\varphi(t)dt=1$

5. $\varphi(x)=\cos x,\quad \varphi(x)-\int_0^{\pi}\left(x^2+t\right)\cos t\varphi(t)dt=\sin x$

## 5.1.2.2. The Method of Fredholm Determinants

The solution of the Fredholm equation of the second kind

$$\varphi(x)-\lambda\int_a^b \mathrm{K}(x,t)\varphi(t)dt=f(x) \qquad (5.1.2.2.1)$$

is given by the formula

$$\varphi(x)=f(x)+\lambda\int_a^b R(x,t;\lambda)f(t)dt \qquad (5.1.2.2.2)$$

where the function $R(x,t;\lambda)$ is called the Fredholm resolvent kernel of equation (5.1.2.21) and is defined by the equation

$$R(x,t;\lambda)=\frac{D(x,t;\lambda)}{D(\lambda)} \qquad (5.1.2.2.3)$$

provided that $D(\lambda)\neq 0$. Here $D(x,t;\lambda)$ and $D(\lambda)$ are power series in $\lambda$

$$D(x,t;\lambda)=\mathrm{K}(x,t)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}B_n(x,t)\lambda^n \qquad (5.1.2.2.4)$$

$$D(\lambda)=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}C_n\lambda^n \qquad (5.1.2.2.5)$$

whose coefficients are given by the formulas

$$B_n(x,t)=\underbrace{\int_a^b \cdots \int_a^b}_{n} \begin{vmatrix} K(x,t) & K(x,t_1) & \cdots & K(x,t_n) \\ K(t_1,t) & K(t_1,t_1) & \cdots & K(t_1,t_n) \\ K(t_2,t) & K(t_2,t_1) & \cdots & K(t_2,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,t) & K(t_n,t_1) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \ldots dt_n \qquad (5.1.2.2.6)$$

and $B_0(x,t)=K(x,t)$

$$C_n=\underbrace{\int_a^b \cdots \int_a^b}_{n} \begin{vmatrix} K(t_1,t_1) & K(t_1,t_2) & \cdots & K(t_1,t_n) \\ K(t_2,t_1) & K(t_2,t_2) & \cdots & K(t_2,t_n) \\ K(t_3,t_1) & K(t_3,t_2) & \cdots & K(t_3,t_n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ K(t_n,t_1) & K(t_n,t_2) & \cdots & K(t_n,t_n) \end{vmatrix} dt_1 \ldots dt_n \qquad (5.1.2.2.7)$$

The function $D(x,t;\lambda)$ is called the Fredholm minor, and $D(\lambda)$ the Fredholm determinant. When the kernel $K(x,t)$ is bounded or the integral

$$\int_a^b\int_a^b K^2(x,t)\,dx\,dt$$

has a finite value, the series (5.1.2.2.4) and (5.1.2.2.5) coverage for all values of $\lambda$ and hence are entire analytic functions of $\lambda$.
The resolvent kernel

$$R(x,t;\lambda)=\frac{D(x,t;\lambda)}{D(\lambda)}$$

is an analytic function of $\lambda$ which are zeros of the function $D(\lambda)$. The latter are the poles of the resolvent kernel $R(x,t;\lambda)$.

Example: Using Fredholm determinants, find the resolvent kernel of the kernel $K(x,t)=xe^t; a=0, b=1$.
Solution. We have $B_0(x,t)=xe^t$. Further

$$B_1(x,t)=\int_0^1 \begin{vmatrix} xe^t & xe^{t_1} \\ t_1e^t & t_1e^{t_1} \end{vmatrix} dt_1 = 0$$

$$B_2(x,t)=\int_0^1\int_0^1\begin{vmatrix} xe^t & xe^{t_1} & xe^{t_2} \\ t_1e^t & t_1e^{t_1} & t_1e^{t_2} \\ t_2e^t & t_2e^{t_1} & t_2e^{t_2}\end{vmatrix}dt_1dt_2=0$$

Since the determinants under the integral sign are zero. It is obvious that all subsequent $B_n(x,t)=0$. Find the coefficients $C_n$.

$$C_1=\int_0^1 K(t_1,t_1)dt_1=\int_0^1 t_1e^{t_1}dt_1=1$$

$$C_2=\int_0^1\int_0^1\begin{vmatrix} t_1e^{t_1} & t_1e^{t_2} \\ t_2e^{t_1} & t_2e^{t_{21}}\end{vmatrix}dt_1dt_2=0$$

Obviously all subsequent $C_n$ are also equal to zero.

In our case, by formulas (5.1.2.2.4) and (5.1.2.2.5) we have

$$D(x,t;\lambda)=K(x,t)=xe^t;\ D(\lambda)=1-\lambda$$

Thus,

$$R(x,t;\lambda)=\frac{D(x,t;\lambda)}{D(\lambda)}=\frac{xe}{1-\lambda}$$

Let us apply the result obtained to solving the integral equation

$$\varphi(x)-\lambda\int_0^1 xe^t\varphi(t)dt=f(x)\quad(\lambda\neq 1)$$

By formula (5.1.2.2.2)

$$\varphi(x)=f(x)+\lambda\int_0^1\frac{xe^t}{1-\lambda}f(t)dt$$

In particular for $f(x)=e^{-x}$ we get

$$\varphi(x)=e^{-x}+\frac{\lambda}{1-\lambda}x$$

## 5.1.2.2.1 Self Assessment Questions :

1. $K(x,t)=2x-t;\qquad 0\le x\le 1,\quad 0\le t\le 1$

2. $K(x,t)=x^2t-xt^2;\qquad 0\le x\le 1,\quad 0\le t\le 1$

3. $K(x,t) = \sin x \cos t;$ $\quad 0 \le x \le 2\pi\,,\quad 0 \le t \le 2\pi$

4. $K(x,t) = \sin x - \sin t;$ $\quad 0 \le x \le 2\pi\,,\quad 0 \le t \le 2\pi$

Practically speaking, only in very rare cases is it possible to compute the coefficients $B_n(x,t)$ and $C_n$ of the series (5.1.2.2.4) and (5.1.2.2.5) from formulas (5.1.2.2.6) and (5.1.2.2.7), but from these formulas it is possible to obtain the following recursion relations:

$$B_n(x,t) = C_n K(x,t) - n\int_a^b K(x,s) B_{n-1}(s,t)\,ds \qquad (5.1.2.2.8)$$

$$C_n = \int_a^b B_{n-1}(s,s)\,ds \qquad (5.1.2.2.9)$$

Knowing that the coefficient $C_0 = 1$ and $B_0(x,t) = K(x,t)$, we can use formulas (5.1.2.2.9) and (5.1.2.2.8) to find, in succession $C_1, B_1(x,t), B_2(x,t), C_3$ and so on.

Example: Using formulas (5.1.2.2.8) and (5.1.2.2.9) find the resolvent kernel for the kernel $K(x,t) = x - 2t$, where $0 \le x \le 1$, $0 \le t \le 1$

Solution: We have $C_0 = 1, B_0(x,t) = x - 2t$. Using formula (5.1.2.2.9) we find

$$C_1 = \int_0^1 (-s)\,ds = -\frac{1}{2}$$

By formula (5.1.2.2.8) we get

$$B_1(x,t) = -\frac{x-2t}{2} - \int_0^1 (x-2s)(s-2t)\,ds = -x - t + 2xt + \frac{2}{3}$$

We further obtain

$$C_2 = \int_0^1 \left(-2s + 2s^2 + \frac{2}{3}\right) ds = \frac{1}{3}$$

$$B_2(x,t) = -\frac{x-2t}{3} - 2\int_0^1 (x-2s)\left(-s - t + 2st + \frac{2}{3}\right) ds = 0$$

$$C_3 = C_4 = \ldots.0, B_3(x,t) = B_4(x,t) = \ldots = 0$$

Hence,

$$D(\lambda)=1+\frac{\lambda}{2}+\frac{\lambda^2}{6};$$

$$D(x,t;\lambda)=x-2t+\left(x+t-2xt-\frac{2}{3}\right)\lambda$$

The resolvent kernel of the given kernel is

$$R(x,t;\lambda)=\frac{x-2t+\left(x+t-2xt-\frac{2}{3}\right)\lambda}{1+\frac{\lambda}{2}+\frac{\lambda^2}{6}}$$

## 5.1.2.2.2. Self Assessment Questions:

Using the recursion relations (5.1.2.2.8) and (5.1.2.2.9), find the resolvent kernels of the following kernels.

1. $K(x,t)= x + t + 1; \qquad -1 \le x \le 1, \quad -1 \le t \le 1$

2. $K(x,t)=1+3xt; \qquad 0 \le x \le 1, \quad 0 \le t \le 1$

3. $K(x,t)= 4xt - x^2; \qquad 0 \le x \le 1, \quad 0 \le t \le 1$

4. $K(x,t)= e^{x-t}; \qquad 0 \le x \le 1, \quad 0 \le t \le 1$

5. $K(x,t)= \sin(x+t); \qquad 0 \le x \le 2\pi, \quad 0 \le t \le 2\pi$

Using the resolvent kernel, solve the following integral equations:

1. $\varphi(x)-\lambda\int_0^{2\pi}\sin(x+t)\varphi(t)dt=1$

2. $\varphi(x)-\lambda\int_0^{1}(2x-t)\varphi(t)dt=\frac{x}{6}$

3. $\varphi(x)-\int_0^{2\pi}\sin x\cos t\,\varphi(t)dt=\cos 2x$

4. $\varphi(x)+\int_0^1 e^{x-t}\varphi(t)dt=e^x$

5. $\varphi(x)-\lambda\int_0^1 (4xt-x^2)\varphi(t)dt=x$

## 5.1.2.3. Iterated Kernels. Constructing the Resolvent Kernel with the Aid of Iterated Kernels

Suppose we have a Fredholm integral equation

$$\varphi(x)-\lambda\int_a^b \mathrm{K}(x,t)\varphi(t)dt=f(x) \qquad (5.1.2.3.1)$$

As in the case of the Volterra equations, the integral equation (5.1.2.3.1) may be solved by the method of successive approximations. To do this, put

$$\varphi(x)=f(x)+\sum_{n=1}^{\infty}\psi_n(x)\lambda^n \qquad (5.1.2.3.2)$$

where the $\psi_n(x)$ are determined from the formulas

$$\psi_1(x)=\int_a^b \mathrm{K}(x,t)f(t)dt$$

$$\psi_2(x)=\int_a^b \mathrm{K}(x,t)\psi_1(t)dt=\int_a^b \mathrm{K}_2(x,t)f(t)dt$$

$$\psi_3(x)=\int_a^b \mathrm{K}(x,t)\psi_{21}(t)dt=\int_a^b \mathrm{K}_3(x,t)f(t)dt$$

and so on.
Here

$$\mathrm{K}_2(x,t)=\int_a^b \mathrm{K}(x,z)\mathrm{K}_1(z,t)dz$$

$$\mathrm{K}_3(x,t)=\int_a^b \mathrm{K}(x,z)\mathrm{K}_2(z,t)dz$$

and generally,

$$K_n(x,t)=\int_a^b K(x,z)K_{n-1}(z,t)dz \qquad (5.1.2.3.3)$$

n=2,3,... and $K_1(x,t)=K(x,t)$ The functions $K_n(x,t)$ determined from formulas (5.1.2.3.3) are called iterated kernels. For them, the following relation holds:

$$K_n(x,t)=\int_a^b K_m(x,s)K_{n-m}(s,t)ds \qquad (5.1.2.3.4)$$

where m is any natural number less than n.

The resolvent kernel of the integral equation (5.1.2.3.1) is determined in terms of iterated kernels by the formula

$$R(x,t;\lambda)=\sum_{n=1}^{\infty}K_n(x,t)\lambda^{n-1} \qquad (5.1.2.3.5)$$

where the series on the right is called the Neumann series of the kernel $K(x,t)$. It converges for

$$|\lambda|<\frac{1}{B} \qquad (5.1.2.3.6)$$

where $B=\sqrt{\int_a^b\int_a^b K^2(x,t)dx\,dt}$

The solution of the Fredholm equation of the second kind (5.1.2.3.1) is expressed by the formula is expressed by the formula

$$\varphi(x)=f(x)+\lambda\int_a^b R(x,t;\lambda)f(t)dt \qquad (5.1.2.3.7)$$

The boundary (5.1.2.3.6) is essential for convergence of the series (5.1.2.3.5). However a solution of equation (5.1.2.3.1) can exist for values of $|\lambda|>\frac{1}{B}$ as well.

Let us consider an example:

$$\varphi(x)-\lambda\int_0^1\varphi(t)dt=1 \qquad (5.1.2.3.8)$$

Here $K(x,t)=1$, and hence

$$B^2=\int_0^1\int_0^1 K^2(x,t)dx\,dt=\int_0^1\int_0^1 dx\,dt=1$$

Thus the condition (5.1.2.3.6) gives that the series (5.1.2.3.5) converges for $|\lambda|<1$

Solving equation (5.1.2.3.8) as an equation with a degenerate kernel, we get $(1-\lambda)C = 1$, where $C = \int_0^1 \varphi(t)dt$. For $|\lambda| > 1$, hence for $\lambda = 1$ the integral equation (5.1.2.3.8) does not have any solution. From this it follows that in a circle of radius greater than unity, successive approximations cannot converge for equation (5.1.2.3.8). However, equation (5.1.2.3.8) is solvable for $|\lambda| < 1$. Indeed, if $\lambda \neq 1$, then the function $\varphi(x) = \dfrac{1}{1-\lambda}$ is a solution of the given equation. This may readily be verified by direct substitution.

For some Fredholm equations the Neumann series (5.1.2.3.5) converges for the resolvent kernel for any values of $\lambda$. Let us demonstrate this fact.

Suppose we have two kernels: $K(x,t)$ and $L(x,t)$. We shall call these kernels orthogonal if the following two conditions are fulfilled for any admissible values of x and t.

$$\int_a^b \mathrm{K}(x,z)L(z,t)dz = 0, \quad \int_a^b L(x,z)K(z,t)dz = 0 \qquad (5.1.2.3.9)$$

Example: The kernels $K(x,t) = xt$ and $L(x,t) = x^2t^2$ are orthogonal on [-1,1].

Indeed,

$$\int_{-1}^{1}(xz)(z^2t^2)dz = xt^2\int_{-1}^{1} z^3 dz = 0,$$

$$\int_{-1}^{1}(x^2z^2)(zt)dz = x^2t\int_{-1}^{1} z^3 dz = 0,$$

There exist kernels which are orthogonal to themselves. For such kernels $K_2(x,t) = 0$ where $K_2(x,t)$ is the second iterated kernel. It is obvious that in this case all subsequent iterated kernels are also equal to zero and the resolvent kernel coincides with the kernel $K(x,t)$

Example: $K(x,t) = \sin 9x - 2t);0 \le x \le 2\pi, 0 \le t \le 2\pi$

We have

$$\int_0^{2\pi} \sin(x-2z)\sin(z-2t)dz$$

$$= \frac{1}{2}\int_0^{2\pi}[\cos(x+2t-3z)-\cos(x-2t-z)]dz =$$

$$= \frac{1}{2}\left[-\frac{1}{3}\sin(x+2t-3z)+\sin(x-2t-z)\right]\Bigg|_{z=0}^{z=2\pi} = 0$$

Thus in this case the resolvent kernel of the kernel is equal to the kernel itself:

$$R(x,t;\lambda) \equiv \sin(x-2t)$$

so that the Neumann series (5.1.2.3.5) consists of one term and obviously, converges for any $\lambda$.

The iterated kernels $K_n(x,t)$ can be expressed directly in terms of the given kernel $K(x,t)$ by the formula

$$K_n(x,t) = \int_a^b\int_a^b\ldots\int_a^b K(x,s_1)K(s_1,s_2)\ldots$$

$$K(s_{n-1},t)ds_1\ldots ds_{n-1} \qquad (5.1.2.3.10)$$

All iterated kernels $K_n(x,t)$ beginning with $K_2(x,t)$ will be continuous functions in the square $a \le x \le b, a \le t \le b$ if the initial kernel $K(x,t)$ is quadratically sum able in this square.

**Example:** Find the iterated kernels for the kernel $K(x,t) = x-t$ if a=0,b=1.

Solution: Using formula (5.1.2.3.2) we find in succession:

$$K_1(x,t) = x-t$$

$$K_2(x,t) = \int_0^1 (x-s)(s-t)ds = \frac{x+t}{2} - xt - \frac{1}{3}$$

$$K_3(x,t) = \int_0^1 (x-s)\left(\frac{s+t}{2} - st - \frac{1}{3}\right)ds = -\frac{x-t}{12}$$

$$K_4(x,t) = -\frac{1}{12}\int_0^1 (x-s)(s-t)ds = -\frac{1}{12}K_2(x,t) = -\frac{1}{12}\left(\frac{x+t}{2} - xt - \frac{1}{3}\right)$$

$$K_5(x,t) = -\frac{1}{12}\int_0^1 (x-s)\left(\frac{x+t}{2} - xt - \frac{1}{3}\right)ds = -\frac{1}{12}K_3(x,t) = \frac{x-t}{12^2}$$

$$K_6(x,t)=\frac{1}{12^2}\int_0^1(x-s)(s-t)ds=\frac{K_2(x,t)}{12^2}=\frac{1}{12^2}\left(\frac{x+t}{2}-xt-\frac{1}{3}\right)$$

From this it follows that iterated kernels are of the form:

(1) for n=2k-1

$$K_{2k-1}(x,t)=\frac{(-1)^k}{12^{k-1}}(x-t)$$

(2) for n=2k, $K_{2k}(x,t)=\frac{(-1)^{k-1}}{12^{k-1}}\left(\frac{x+t}{2}-xt-\frac{1}{3}\right)$

where k=1,2,3,…

## 5.1.2..3.1. Self Assessment Questions:

1. $K(x,t)=\sin(x-t);\quad a=0,\quad b=\frac{\pi}{2}\quad (n=2,3)$
2. $K(x,t)=(x-t)^2;\quad a=-1,\quad b=1\quad (n=2,3)$
3. $K(x,t)=x+\sin t;\quad a=-\pi\ ,\quad b=\pi$
4. $K(x,t)=xe^t;\quad a=0,\quad b=1$

## 5.1.3. Integral Equations with Degenerate Kernels.

The kernel $K(x,t)$ of a Fredholm integral equation of the second kind is called degenerate if it is the sum of a finite number of products of functions of x alone by functions of t alone; i.e., if it is of the form

$$K(x,t)=\sum_{k=1}^{n}a_k(x)b_k(t)\qquad(5.1.3.1)$$

We shall consider the functions $a_k(x)$ and $b_k(t)$ (k=1,2,…,n) continuous in the basic square $a\le x,t\le b$ and linearly independent. The integral equation with degenerate kernel (5.1.3.1)

$$\varphi(x)-\lambda\int_a^b\left[\sum_{k=1}^{n}a_k(x)b_k(t)\right]\varphi(t)dt=f(x)\qquad(5.1.3.2)$$

is solved in the following manner.

Rewrite (5.1.3.2) as

$$\varphi(x)=f(x)+\lambda\sum_{k=1}^{n}a_k(x)\int_a^b b_k(t)\varphi(t)dt\qquad(5.1.3.3)$$

and introduce the notation

$$\int_a^b b_k(t)\varphi(t)dt = C_k \qquad (k = 1,2,\dots,n) \qquad (5.1.3.4)$$

Then (5.1.3.3) becomes

$$\varphi(x) = f(x) + \lambda \sum_{k=1}^{n} C_k a_k(x) \qquad (5.1.3.5)$$

where $C_k$ are unknown constants, since the function $\varphi(x)$ is unknown.

Thus, the solution of an integral equation with degenerate kernel reduces to finding the constants $C_k(k = 1,2,...n)$ Putting the expression (5.1.3.5) into the integral equation (5.1.3.2), we get (after simple manipulations)

$$\sum_{m=1}^{n}\left\{C_m - \int_a^b b_m(t)\left[f(t) + \lambda \sum_{k=1}^{n} C_k a_k(t)\right]dt\right\} a_m(x) = 0$$

Whence it follows, by virtue of the linear independence of the functions $a_m(x)(m = 1,2,...,n)$, that

$$C_m - \int_a^b b_m(t)\left[f(t) + \lambda \sum_{k=1}^{n} C_k a_k(t)\right]dt = 0$$

or

$$C_m - \lambda \sum_{k=1}^{n} C_k \int_a^b a_k(t)b_m(t)dt = \int_a^b b_m(t)f(t)dt \qquad (m = 1,2,\dots,n)$$

For sake of brevity, we introduce the notations

$$a_{km} = \int_a^b a_k(t)b_m(t)dt, \quad f_m = \int_a^b b_m(t)f(t)dt$$

and find that

$$C_m - \lambda \sum_{k=1}^{n} a_{km} C_k = f_m \qquad (m = 1,2,\dots,n)$$

or, in expanded form

$$\left.\begin{array}{l} (1-\lambda\, a_{11})C_1 - \lambda\, a_{12}C_2 - \dots - \lambda\, a_{1n}C_n = f_1 \\ \lambda\, a_{21}C_1 + (1-\lambda\, a_{22})C_2 - \dots - \lambda\, a_{2n}C_n = f_2 \\ \cdots\cdots\cdots\cdots\cdots \\ \lambda\, a_{n1}C_1 - \lambda\, a_{n2}C_2 - \dots + (1-\lambda\, a_{nn})C_n = f_1 \end{array}\right\} \qquad (5.1.3.6)$$

For finding the unknowns $C_k$ we have a linear system of n algebraic equations in n unknowns. The determinant of this system is

$$\Delta(\lambda)=\begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \dots & -\lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ -\lambda a_{n1} & -\lambda a_{n2} & \dots & 1-\lambda' a_{nn} \end{vmatrix} \qquad (5.1.3.7)$$

If $\nabla\lambda \neq 0$ then the system (5.1.3.6) has a unique solution $C_1$, $C_2$,....$C_n$ which is obtained from Crammers formulas.

$$C_k=\frac{1}{\Delta(\lambda)}\begin{vmatrix} 1-\lambda a_{11} & \dots & -\lambda a_{1k-1}f_1 & -\lambda a_{1k+1} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & \dots & -\lambda a_{2k-1}f_2 & -\lambda a_{2k+1} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -\lambda a_{n1} & \dots & -\lambda a_{nk-1}f_n & -\lambda a_{nk+1} & \dots & 1-\lambda a_{nn} \end{vmatrix} \qquad (5.1.3.8)$$

The solution of the integral equation (5.1.3.2) is the function $\varphi(x)$ defined by the equality

$$\varphi(x)=f(x)+\lambda\sum_{k=1}^{n}C_k a_k(x)$$

where the coefficients $C_k$ (k=1,2,3,...n) are determined from formulas (5.1.3.8)

## 5.1.3.1. Self Assessment Questions:

1. $\varphi(x)-4\int_0^{\frac{\pi}{2}}\sin^2 x\varphi(t)dt=2x-\pi$

2. $\varphi(x)-\int_{-1}^{1}e^{\arcsin x}\varphi(t)dt=\tan x$

3. $\varphi(x)-\lambda\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\tan t\varphi(t)dt=\cot x$

4. $\varphi(x)-\lambda\int_0^1\cos(q\ln t)\varphi(t)dt=1$

5. $\varphi(x)-\int_0^1\arccos t\varphi(t)dt=\frac{1}{\sqrt{1-x^2}}$

## 5.1.4. Characteristic Number and Eigen functions

The homogeneous Fredholm integral equation of the second kind

$$\varphi(x)-\lambda\int_a^b K(x,t)\varphi(t)dt=0 \qquad (5.1.4.1)$$

always has the obvious solution $\phi(x)\equiv 0$ which is called the zero solution.

The values of the parameter $\lambda$ for which this equation has non zero solutions $\phi(x)\neq 0$ are called characteristic numbers of the equation (5.1.4.1) or of the kernel K(x,t) and every non zero solution of this equation is called an eigen function corresponding to the characteristic number $\lambda$.

The number $\lambda$=0 is not a characteristic number since for $\lambda$=0 it follows from (5.1.4.1) that . $\phi(x)\equiv 0$.

If the kernel K(x,t) is continuous in the square $\Omega\{a\le x,t\le b\}$ or is quadratically summable in $\Omega$ and the numbers a and b are finite, then to every characteristic number $\lambda$ there corresponds a finite number of linearly independent eigen functions; the number of such functions is called the index of the characteristic number. Different characteristic number can have different indices.

For an equation with degenerate kernel

$$\varphi(x)-\lambda\int_a^b\left[\sum_{k=1}^n a_k(x)b_k(t)\right]\varphi(t)dt=0 \qquad (5.1.4.2)$$

the characteristic numbers are roots of the algebraic equation

$$\Delta(\lambda)=\begin{vmatrix} 1-\lambda a_{11} & -\lambda a_{12} & \dots & -\lambda a_{1n} \\ -\lambda a_{21} & 1-\lambda a_{22} & \dots & -\lambda a_{2n} \\ \dots & \dots & \dots & \dots \\ -\lambda a_{n1} & -\lambda a_{n2} & \dots & 1-\lambda a_{nn} \end{vmatrix}=0 \qquad (5.1.4.3)$$

the degree of which is $\rho\le n$. Here $\Delta(\lambda)$ is the determinant of the linear homogeneous system

$$(1-\lambda a_{11})C_1-\lambda a_{12}C_2-\dots-\lambda a_{1n}C_n=0$$

$$\lambda a_{21}C_1+(1-\lambda a_{22})C_2-\dots-\lambda a_{2n}C_n=0 \qquad (5.1.4.4)$$

.....................................................

$$\lambda a_{n1}C_1-\lambda a_{n2}C_2-\dots+(1-\lambda a_{nn})C_n=0$$

where the quantities $a_{mk}$ and $C_m$ (k, m=1,2,….,n) have the same meaning as in the preceding section.

If equation (5.1.4.3) has p roots $(1\le p\le n)$ then the integral equation (5.1.4.2) has p characteristic numbers; to each characteristic number $\lambda_m (m = 1,2.., p)$ there corresponds a non zero solution

$$C_1^{(1)}, C_2^{(1)}, \ldots, C_n^{(1)} \to \lambda_1$$

$$C_1^{(2)}, C_2^{(2)}, \ldots, C_n^{(2)} \to \lambda_2 \qquad \ldots\ldots\ldots\ldots$$

$$C_1^{(p)}, C_2^{(p)}, \ldots, C_n^{(p)} \to \lambda_p$$

of the system (5.1.4.4). The non zero solution of the integral equation (5.1.4.2) corresponding to these solutions, i.e., the eigen functions, will be of the form

$$\varphi_1(x) = \sum_{k=1}^{n} C_k^{(1)} a_k(x),\ \varphi_2(x) = \sum_{k=1}^{n} C_k^{(2)} a_k(x),\ \ldots,$$

$$\varphi_p(x) = \sum_{k=1}^{n} C_k^{(p)} a_k(x)$$

An integral equation with degenerate kernel has at most n characteristic numbes and (corresponding to them) eigen functions. In the case of an arbitrary (non degenerate) kernel, he characteristic numbers are zeros of the Fredholm determinant $D(\lambda)$, e.e., are poles of he resolvent kernel $R(x,t;\lambda)$. It then follows in particular, that the Volterra integral equation $\varphi(x) - \lambda \int_0^x K(x,t)\varphi(t)dt = 0$ where $K(x,t) \in L_2(\Omega_0)$ has no charteristic numbers (for it, $D(\lambda) = e^{-A_1\lambda}$

Example: Find the characteristic numbers and eigen functions of he integral equation

$$\varphi(x) - \lambda \int_0^\pi (\cos^2 x \cos 2t + \cos 3x \cos^3 t)\varphi(t)dt = 0$$

Solution. We have

$$\varphi(x) = \lambda \cos^2 x \int_0^\pi \varphi(t)\cos 2t\, dt + \lambda \cos 3x \int_0^\pi \varphi(t)\cos^3 t dt$$

Introducing the notations

$$C_1 = \int_0^\pi \varphi(t)\cos 2t\, dt, \qquad C_2 = \int_0^\pi \varphi(t)\cos^3 t\, dt \qquad (1)$$

We get

$$\varphi(x) = C_1\lambda\cos^2 x + C_2\lambda\cos 3x \qquad (2)$$

Substituting (2) into (1) we obtain a linear system of homogeneous equations:

$$C_1\left(1-\lambda\int_0^\pi \cos^2 t\cos 2t\,dt\right) - C_2\lambda\int_0^\pi \cos 3t\cos 2t\,dt = 0$$

$$-C_1\lambda\int_0^\pi \cos^5 t\,dt + C_2\left(1-\lambda\int_0^\pi \cos^3 t\cos 3t\,dt\right) \qquad (3)$$

But since

$$\int_0^\pi \cos^2 t\cos 2t\,dt = \frac{\pi}{4}, \qquad \int_0^\pi \cos 3t\cos 2t\,dt = 0$$

$$\int_0^\pi \cos^5 t\,dt = 0, \qquad \int_0^\pi \cos^3 t\cos 3t\,dt = \frac{\pi}{8}$$

It follows that system (3) takes the form

$$\left(1-\frac{\lambda\pi}{4}\right)C_1 = 0$$

$$\left(1-\frac{\lambda\pi}{8}\right)C_2 = 0 \qquad (4)$$

The equation for finding characteristic numbers will be

$$\begin{vmatrix} 1-\frac{\lambda\pi}{4} & 0 \\ 0 & 1-\frac{\lambda\pi}{8} \end{vmatrix} = 0$$

The characteristic numbers are $\lambda_1 = \frac{4}{\pi}$, $\lambda_2 = \frac{8}{\pi}$

For $\lambda = \frac{4}{\pi}$, system (4) becomes

$$\begin{cases} 0.C_1 = 0 \\ \frac{1}{2}.C_2 = 0 \end{cases}$$

whence $C_2$=0,$C_1$ is arbitrary. The eigen function will be $\phi_1(x) = \cos^2 x$.

For $\lambda = \frac{8}{\pi}$, system (4) is of the form

$$\begin{cases}(-1).C_1 = 0\\ 0.C_2 = 0\end{cases}$$

whence $C_1$=0, $C_2$ is arbitrary and hence, the eigen function will be $\phi_2(x) = C_2\lambda\cos 3x$,

or assuming $C_2\lambda = 1$, we get $\phi_2(x) = \cos 3x$,

Thus, the characteristic numbers are

$$\lambda_1 = \frac{4}{\pi}, \qquad \lambda_2 = \frac{8}{\pi}$$

and the corresponding eigen functions are

$$\phi_1(x) = \cos^2 x, \qquad \phi_2(x) = \cos 3x$$

A homogenous Fredholm integral equation may, generally, have no characteristic numbers and eigen functions. Or it may not have any real characteristic numbers and eigen functions.

## 5.1.4.1. Self Assessment Questions:

Find the characteristic numbers and eigen functions for the following homogeneous integral equations with degenerate kernels:

1. $\varphi(x) - \lambda \int_0^{\frac{\pi}{4}} \sin^2 x \varphi(t)dt = 0$

2. $\varphi(x) - \lambda \int_0^{2\pi} \sin x \cos t \varphi(t)dt = 0$

3. $\varphi(x) - \lambda \int_0^{2\pi} \sin x \sin t \varphi(t)dt = 0$

4. $\varphi(x) - \lambda \int_0^{\pi} \cos(x+t)\varphi(t)dt = 0$

5. $\varphi(x) - \lambda \int_0^{1} (45x^2 \ln t - 9t^2 \ln x)\varphi(t)dt = 0$

## 5.1.5. Solution of homogeneous integral equations with degenerate kernel

The homogonous integral equation with degenerate kernel

$$\varphi(x)-\lambda\int_a^b\left[\sum_{k=1}^{n}a_k(x)b_k(t)\right]\varphi(t)dt=0 \qquad (5.1.5.1)$$

where the parameter $\lambda$ is not its characteristic number (i.e., $\Delta\lambda \neq 0$) has a unique zero solution: $\phi(x)=0$. But if $\lambda$ is a characteristic number, then besides the zero solution, equation (5.1.5.1) also has non zero solutions- the eigen functions which correspond to that characteristic number. The general solution of the homogonous equation (5.1.5.1) is obtained as a linear combination of these eigen functions.

Example: Solve the equation

$$\varphi(x)-\lambda\int_0^{\pi}\left(\cos^2 x\cos 2t+\cos^3 t\cos 3x\right)\varphi(t)dt=0$$

Solution. The characteristic numbers of this equation are $\lambda_1=\frac{4}{\pi}$, $\lambda_2=\frac{8}{\pi}$; the corresponding eigen functions are of the form

$$\phi_1(x)=\cos^2 x, \phi_2(x)=\cos 3x$$

The general solution of the equation is

$$\varphi(x)=C\cos^2 x \qquad \text{if} \quad \lambda=\frac{4}{\pi}$$

$$\varphi(x)=C\cos 3x \qquad \text{if} \quad \lambda=\frac{8}{\pi}$$

$$\varphi(0)=0 \qquad \text{if} \quad \lambda\neq\frac{4}{\pi}, \lambda\neq\frac{8}{\pi}$$

Where C is an arbitrary constant. The last zero solution is obtained from the general solutions for C=0.

## 5.1.5.1. Self Assessment Questions:

1. $\varphi(x)-\lambda\int_0^{\pi}\cos(x+t)\varphi(t)\,dt=0$

2. $\varphi(x)-\lambda\int_0^{1}\arccos x\varphi(t)\,dt=0$

3. $\varphi(x)-2\int_0^{\pi/4}\frac{\varphi(t)}{1+\cos 2t}\,dt=0$

4. $\varphi(x)-\frac{1}{4}\int_{-2}^{2}|x|\varphi(t)\,dt=0$

5. $\varphi(x)+6\int_{0}^{1}(x^2-2xt)\varphi(t)\,dt=0$

## 5.1.6. Nonhomogenous Symmetric equation

The non homogonous Fredholm integral equation of the second kind

$$\phi(x)-\lambda\int_a^b K(x,t)\varphi(t)dt = f(x) \qquad (5.1.6.1)$$

is called symmetric if its kernel K(x,t) is symmetric K(x,t)=K(t,x). If f(x) is continuous and the parameter $\lambda$ does not coincide with the characteristic numbers $\lambda_n(n=1,2,...)$ of the corresponding homogonous integral equation

$$\phi(x)-\lambda\int_a^b K(x,t)\varphi(t)dt = 0 \qquad (5.1.6.2)$$

then equation (5.1.6.1) has a unique continuous solution, which is given by the formula

$$\varphi(x)=f(x)-\lambda\sum_{n=1}^{\infty}\frac{a_n}{\lambda-\lambda_n}\varphi_n(x) \qquad (5.1.6.3)$$

where $\phi_n(x)$ are eigen functions of equation (5.1.6.2),

$$a_n=\int_a^b f(x)\phi_n(x)dx \qquad (5.1.6.4)$$

The series on the right side of formula (5.1.6.3) converges absolutely and uniformly in the square $a \le x,t \le b$.

But if the parameter $\lambda$ coincides with one of the characteristic numbers say $\lambda=\lambda_k$ of index q (multiplicity of the number $\lambda_k$) then equation (5.1.6.1) will not, generally speaking, have any solutions. Solutions exist if an only if the q conditions are fulfilled:

$$(f,\varphi_m)=0 \text{ or } \int_a^b f(x)\varphi_m(x)dx=0 \qquad (5.1.6.5)$$

(m=1,2,…q)

that is, if the function f(x) is orthogonal to all eigen functions belonging to the characteristic number . $\lambda_k$ In this case equation (5.1.6.1) has an infinity of solutions which contain q arbitrary constants and are given by the formula

$$\varphi(x) = f(x) - \lambda \sum_{n=1}^{\alpha} \frac{a_n}{\lambda - \lambda_n} \varphi_n(x) + C_1\varphi_1(x) + C_2\varphi_2(x) + \ldots + C_q\varphi_q(x) \quad (5.1.6.6)$$

where $C_1, C_2, \ldots, C_q$ are arbitrary constants.
In the case of the degenerate kernel

$$K(x,t) = \sum_{k=1}^{m} a_k(x) b_k(t)$$

formulas (5.1.6.3) and (5.1.6.6) will contain finite sums in place of series in their right hand members.
When the right hand side of equation (5.1.6.1) i.e., the function f(x) is orthogal to all eigen functions $\phi_n(x)$ of equation (5.1.6.2), the function itself will be a solution of equation (5.1.6.1): $\phi(x) = f(x)$.

**Example**: Solve the equation

$$\phi(x) - \lambda \int_0^1 K(x,t)\varphi(t)dt = x \quad (1)$$

where $K(x,t) = \begin{cases} x(t-1) & if\, 0 \le x \le t \\ t(x-1) & if\, t \le x \le 1 \end{cases}$

**Solution**. The characteristic numbers and their associated eigen functions are of the form

$$\lambda_n = -\pi^2 n^2, \varphi n(x) \sin \pi nx, n = 1,2,\ldots$$

If $\lambda \neq \lambda_n$, then

$$\varphi(x) = x - \lambda \sum_{n=1}^{\alpha} \frac{a_n}{\lambda + n^2\pi^2} \sin \pi nx \quad (2)$$

will be a solution of equation (1). We find the Fourier coefficients $a_n$ of the right side of the equation:

$$a_n = \int_0^1 x \sin n\pi x = \int_0^1 x d\left(-\frac{\cos n\pi x}{n\pi}\right) = \frac{(-1)^{n+1}}{n\pi}$$

Substituting into (5.1.6.2) we get

$$\varphi(x) = x - \frac{\lambda}{\pi} \sum_{n=1}^{\alpha} \frac{(-1)^{n+1}}{n(\lambda + n^2\pi^2)} \sin \pi nx$$

for $\lambda = -n^2\pi^2$ equation (1) has no solutions since

$$a_n = \frac{(-1)^{n+1}}{n\pi} \neq 0$$

## 5.1.6.1. Self Assessment Questions:

Solve the following non homogonous symmetric integral equations.

1. $$\varphi(x) - \frac{\pi^2}{4}\int_0^1 K(x,t)\varphi(t)dt = \frac{x}{2},$$

$$K(x,t) = \begin{cases} \dfrac{x(2-t)}{2} & 0 \le x \le t, \\ \dfrac{t(2-x)}{2} & t \le x \le 1 \end{cases}$$

2. $$\phi(x) + \int_0^1 K(x,t)\varphi(t)dt = xe^x$$

$$K(x,t) = \begin{cases} \dfrac{\sinh x \sinh(t-1)}{\sinh l} & 0 \le x \le t, \\ \dfrac{\sinh t \sinh(x-1)}{\sinh l} & t \le x \le 1 \end{cases}$$

## 5.1. 7 Let us sum up

In this unit we have covered the following points

1. We have explained the concept of Fredholm equation
2. We have explained the method of Fredholm determinant, iterated kernels and construction of resolvent kernel with the help of iterated kernels
3. We have also given the idea of integral equation with degenerate kernel, Characteristic numbers and eigen functions.
4. Finally we have explained the solution of homogeneous integral equations with degenerate kernels. Concept of non-homogeneous symmetric equations is also provided.